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Fundamentals of Microcrack Nucleation Mechanics

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SECTION I

Introduction

Previous theoretical [1-3] and experimental studies [4] established the foundation for a correlation between the plane strain fracture toughness K_{IC} and the ultrasonic factors $v_L \beta_\delta / m$ through the interaction of waves with material microstructures, i.e. grain size or second-phase particle spacing depending upon the material system. This suggests a means for ultrasonic evaluation of plane strain fracture toughness. Ultrasonic methods can therefore be used not only in the evaluation of material properties such as moduli and porosity but also in the fracture properties such as crack size and fracture toughness.

There are situations where material properties keep changing. Stress induced deformation during manufacturing processes often cause continued changes in material properties. It is well-known that precipitation, martensitic transformation, void nucleation, etc. often lead to changes in material properties. Fracture toughness enhancement has been observed in a number of ceramic systems due to stress induced transformations e.g. Zirconia (ZrO_2) particles contained in ceramic matrix are observed to transform from a tetragonal to a monoclinic crystal structure at sufficiently high stress environment. Circumferential microcracking may occur and a process zone is usually formed in the vicinity of the stress raiser to reduce crack-tip stress intensity.

Precipitates, martensites and dislocations are typical examples of inhomogeneous inclusions [5,6,27,32], i.e., regions with distributed transformation strains alias eigenstrains. Since the scattered displacements are directly related to the eigenstrains [7,8], the scattered field measured can be used to study the changes caused by the presence of precipitates, martensites, voids, etc.

The purpose of this work is to lay a foundation for the ultrasonic evaluation of microcracking. The work establishes an "average" theorem and wave scattering effects of a microcrack and disbonded microvoid.

SECTION II

Technical Background

Definition of Microcrack Nucleation

A microcrack represents solid-vapor surfaces in a material and it has a dimensional scale comparable to some microstructural features, e.g. the grain size. The use of the term microcrack should not be construed to imply brittle fracture. An microcracking event is obviously the origin of fracture. In what follows, a brief review of different mechanisms of microcracking is outlined.

Microcracking in Polycrystals

In single phase crystals, a microcrack nucleated from defect locations can grow at the competition between dislocation generation and bond rupture of the crack tip. The activation or generation of dislocations that pile up at the crack tip essentially determines whether the crystal is brittle or ductile.

For cleavage-prone polycrystalline materials with brittle particles, such as A533B steel, the brittle fracture proceeds by a mechanism involving a slip band blocked by a carbide. The most heavily strained carbide particles are usually found to be a few grains ahead of the crack-tip. The microcracking event therefore occurs at that location under suitable loading conditions. For a polycrystalline material with a matrix that resists cleavage the microcracking can occur by void nucleation and growth mechanism. Stresses arising due to the incompatibility between the inhomogeneities and the material matrix may cause eventual cracking of the particle (inhomogeneity) or the interface.

Depending upon the "size" of the particles involved, secondary plastic zone near the inhomogeneities may or may not be of importance. In essence, the continuum plasticity models can be used for large

inhomogeneities, $\delta \geq 1 \mu\text{m}$ detail dislocation structures must be considered for the medium size particles, $0.01 \mu\text{m} \leq \delta \leq 1 \mu\text{m}$ and at the decohesion of interface, the size of the plastic zone seems to be dependent on the diameter of particles rather than the strains. Shearing of the weak inhomogeneities may be an important mode of microcracking in the case of very small particles, $\delta \geq 0.01 \mu\text{m}$. It is clear from the discussion above that the microcrack nucleation mechanism involves the situations of either (1) the creation of new surfaces by tension or by shear at a weak second-phase particle or (2) the nucleation and growth of voids at a strong second-phase particle.

Microcracking in Ceramics and High Temperature Materials

The microcrack nucleation is an event of crack initiation. The presence of a single microcrack is usually not of serious consequences. It may, however, grow in size to become a macrocrack. Multiple microcracks may nucleate near the vicinity of a macrocrack to form a microfracture process zone thus enhance the coalescence of microcrack that add to the length of a macrocrack. Voids nucleated along grain boundaries may be flattened out into cracks under favorable conditions. Spacing between the voids, can be strongly dependent both on distance from the crack and on the duration of loading [32]. One form of transformation toughening of ceramics with intercrystalline ZrO_2 dispersions was found to be largely caused by microcrack nucleation and extension. The phenomenon of fracture toughness enhancement was explained by the use of circumferential microcracking model [30] and by a continuum elastic plastic fracture mechanics approach.

Dynamic Models of Microcracking

A simple dynamic model for microcracking was first proposed by Vary [4] where he considered the stress wave energy required to create a microcrack of the diameter same as the second-phase particle. He considered this portion of the energy to be totally consumed by the

particle during its fracture and gave an interrelation between the ultrasonic and fracture toughness factors. A theoretical verification of the model requires information about the scattering of an embedded flat microcrack. In the fracture of ceramic materials, there are cases where the failure prediction requires the direct evaluation of the maximum stress intensity factor by either proof testing and/or acoustic emission.

For the elastic wave scattering from planar cracks was given a starting point for the integral formulation is the representation of the scattered fields in terms of the crack-opening displacement, Δu_j ,

$$u_m^{(s)} = -C_{jkrs} \int_S \sigma_{rm,s'} \Delta u_j n_k dS' \quad (1)$$

Numerous investigators have examined the quasi-static scattering from elliptical and circular planar cracks. Since the crack-opening displacement, Δu_j , is not a known-priori in Eq. (1), approximations can be obtained by using well-known exact solutions for cracks under static loading. The obtained solution are thus appropriate for long wavelength. An estimation procedure for fracture crack parameters was previously given by Budiansky and Rice [11]. On the other hand, Fu, Co and Dzeng [14] presented a theory for elastic wave scattering and crack sizing which is appropriate at wider frequency range. Both of these studies were in the frequency domain. It should be noted that information in the frequency domain and the signals in time domain are related via Fourier transforms. The solution presented here applies to the frequency region corresponding to dimensionless wavenumber ka from zero to about less than 2 .

Plan of Study

The microfracture problems can be analyzed by first computing the appropriate microstress distributions, then estimating the stress

intensity factor K for typical microstructural defects located in the stress field, and finally equating K to the local material resistance to microfracture. The microfracture problem of greatest practical interest concerns the grain shape, size, and relative orientation that dictates the onset condition for boundary microfracture between grains.

As a beginning in this program, the elastic wave scattering from a planar crack under incident plane waves is studied. The specific geometries studied are the penny-shaped circular crack and the elliptical crack. Far-field quantities such as the reduced scattering amplitude that are related to the shape, size and orientation of the crack will be determined. Near field quantities such as the crack size, the stress intensity factors and the crack opening displacements are briefly studied [13,14]. The approach taken is based upon the methods developed in [7,8,12].

As a second part of the work for this program, an average theorem is developed to study the dynamic effective properties of media with randomly distributed inhomogeneities. The approach taken is deterministic in nature and the solution is based upon the elastic wave scattering of a single inhomogeneity, a self-consistency scheme and an averaging theorem over strain energy and kinetic energy [8,15].

SECTION III

Theoretical Studies

The aim in non-destructive evaluation (NDE) by sound and ultrasound is to extract material and fracture properties from pulses sent and received at transducers. The analysis is usually given either in the time domain or the frequency domain. The introduction of the size factor, the wavelength, in the model mechanics problems in NDE, plays the important role of relating the far-field measurable quantities such as phase velocities and attenuation to the near-field physical situation such as local geometric dimensions and elastic properties. This aspect of the analysis thus allows "nondestructive" testing methods to be employed for "experimental" validation of the predictions by the theories.

Solution to the elastic wave scattering due to a single embedded inhomogeneity is available by different methods that are appropriate at different frequency ranges [16,10]. The methods that offer a solution in an analytic form and are useful for inhomogeneous media with multiple components are the longwave approximation, the polarization approach and the extended method of equivalent inclusion [7,8].

There are several averaging schemes or theorems that exist in the literature for finding the dynamic effective moduli and mass density. While some efforts concentrate on the average stress $\bar{\sigma}$ and strain $\bar{\epsilon}$ fields or on the average displacement field \bar{u} , others concentrate on a variational approach. These theories are appropriate mostly at Rayleigh or longwave limits and do not exhibit dispersive effects. Dispersiveness and attenuation are important in dynamic material properties evaluation.

In what follows, the scatter of elastic waves due to a thin flat ellipsoidal inhomogeneity is first studied. Penny-shaped and elliptically-shaped cracks are studied as special cases. Fracture crack

parameters are presented. An average theorem appropriate for dynamic effective mass density and effective moduli is developed via a self-consistent scheme. Effective material properties of two-component media consisted of randomly distributed spheres are given as a special case.

Preliminaries

(1) Displacement field due to the presence of mismatch

The inhomogeneous media considered in this paper are assumed to be consisted of a homogeneous matrix, of elastic moduli ζ^0 and mass density ρ^0 , and a distribution of inhomogeneities with moduli $\zeta^{(r)}$ and mass density $\rho^{(r)}$ occupying regions Ω_r , $r = 1, 2, \dots, n$, Fig. 4. The total displacement field \underline{u} can be separated into two parts,

$$\underline{u} = \underline{u}^{(i)} + \underline{u}^{(m)}, \quad (2)$$

where the superscripts (i) and (m) denote "incident" and "mismatch," respectively. It is clear that when no "mismatch" components, i.e. inhomogeneities, are present, the total displacement field is entirely the same as the "incident" displacement wave field. On the other hand, if there is no incident wave field, the only field that exists is the null field.

(2) Eigenstrains

The eigenstrains ϵ^* which are also termed transformation strains are defined as the part of the total strain ϵ that must be subtracted before the remaining part can be related to stresses σ through Hooke's law,

$$\epsilon_{rs}^* = \epsilon_{rs} - \epsilon_{rs}^e \quad (3)$$

$$\epsilon_{rs}^e = C_{jkrs}^{-1} \sigma_{jk} \quad (4)$$

where C^{-1} are the elastic compliances. The method of equivalent inclusion is a method that allows the inhomogeneous media be replaced by media of homogeneous matrix effective moduli with distributed transformation strains in the regions originally occupied by inhomogeneities, hence

$$\epsilon_{rs}^* = \begin{cases} 0 & \text{in } \Omega_0 \\ \epsilon_{rs}^* & \text{in } \Omega_r, r = 1, 2, \dots, n. \end{cases} \quad (5)$$

For the two problems to be equivalent, the transformation strains must give rise to a field that is exactly the same as that of the "mismatch" field, $u^{(m)}$. This leads to equivalence conditions that insure identical field quantities at any given point be obtained in the two problems[†] [9]:

$$\Delta C_{jkrs} u_{r,s}^{(m)}(\bar{r}) + C_{jkrs} \epsilon_{rs}^{*(1)}(\bar{r}) = -\Delta C_{jkrs} u_{r,s}^{(i)}(\bar{r}), \text{ in } \Omega \quad (6)$$

$$\Delta \rho \omega^2 u_j^{(m)}(\bar{r}) + C_{jkrs} \epsilon_{rs,k}^{*(2)}(\bar{r}) = -\Delta \rho \omega^2 u_j^{(i)}(\bar{r}) \quad \text{in } \Omega \quad (7)$$

There are two types of transformation strains or eigenstrains that arise in elastodynamic situations due to the mismatch in elastic moduli ΔC and mass density $\Delta \rho$. It is often convenient and useful to define associated quantities such as

$$m_{jk} = C_{jkrs} \epsilon_{rs}^{*(1)} \quad (8)$$

[†]The conditions (6,7) are similar to those of Willis (1980) and those of Mura, Proc. Int. Conf. on Mechanical Behavior of Materials, 5, Society of Materials Science, Japan, 12-18 (1972).

$$\pi_j = c_{jkrs} \epsilon_{rs,k}^{*(2)} \quad (9)$$

where m_{jk} and π_j are referred to as moment density tensor and equivalent force or eigenforce, respectively.

(3) Volume average and time average

The volume average and time average of a field quantity say $F(\underline{r}, t)$ is denoted by using brackets $\langle \rangle$, and $\langle \rangle_T$, respectively, and are defined as

$$\langle F(\underline{r}, t) \rangle = \frac{1}{V} \int F(\underline{v}, t) dV \quad (10)$$

$$\langle F(\underline{r}, t) \rangle_T = \frac{1}{T} \int F(\underline{r}, t) dT \quad (11)$$

where V and T stand for volume and time period, respectively.

(4) Volume integrals of an ellipsoid associated with inhomogeneous Helmholtz equation

Volume integrals of an ellipsoid associated with the integration of the inhomogeneous Helmholtz equation are used in this work. The inhomogeneous scalar Helmholtz equation takes the form:

$$\nabla^2 \phi + k^2 \phi = -4\pi\gamma(\underline{r}) \quad (12)$$

where $\gamma(\underline{r})$ is the source distribution or density function, ∇^2 and k are the Laplacian and wavenumber, respectively. A particular solution to Eq. (12) is

$$\phi(\underline{r}) = \int_{\Omega} \gamma(\underline{r}') R^{-1} \exp(ikR) dV', \quad R = |\underline{r} - \underline{r}'| \quad (13)$$

in which $(4\pi R)^{-1}\exp(ikR)$ is the steady state scalar wave Green's function and Ω is the region where the source is distributed. The source distribution function $\gamma(\underline{r})$ can be expanded in basis functions or polynomial form, depending upon the geometry of volumetric region Ω . For an ellipsoidal region, the choice of using a polynomial expansion separates this work from other theories of elastic wave scattering:

$$\gamma(\underline{r}') = (x')^\lambda (y')^\mu (z')^\nu \quad (14)$$

in which λ, μ, ν are integers.

For elastic wave scattering in an isotropic elastic matrix, two types of volume integrals and their derivatives need to be evaluated:

$$\psi(\underline{r}) = \int_{\Omega} R^{-1} \exp(i\alpha R) dV' \quad (15a)$$

$$\psi_k(\underline{r}) = \int_{\Omega} x'_k R^{-1} \exp(i\alpha R) dV' \quad (15b)$$

$$\psi_{k\ell\dots s}(\underline{r}) = \int_{\Omega} x'_k x'_\ell \dots x'_s R^{-1} \exp(i\alpha R) dV' \quad (15c)$$

$$\dots \quad \psi_{,p}(\underline{r}) = \frac{\partial}{\partial x_p} \psi(\underline{r}) \quad (16a)$$

$$\psi_{k,p}(\underline{r}) = \frac{\partial}{\partial x_p} \psi_k(\underline{r}) \quad (16b)$$

\dots
where $\alpha^2 = \omega^2/V_L^2 = \rho\omega^2/(\lambda + 2\mu)$. The other type, the ϕ -integrals, are obtained by replacing α with β in Eqs. (13,14), where $\beta^2 = \rho\omega^2/(m\mu)$. Details of the integration are given in [12]. In this work only limiting value at $r \rightarrow 0$ and $r \rightarrow \infty$ are of interest.

An Isolated Flat Ellipsoidal Crack

(1) Formulation [28,13,8]

Consider the physical problem of an isolated inhomogeneity embedded in an infinite elastic solid which is subjected to a plane time-harmonic incident wave field as depicted in Fig. 1. Replacing the inhomogeneity with the same material as that of the surrounding medium, with moduli $C_{jkr s}$ and mass density ρ , and include in this region a distribution of eigenstrains and eigenforces, the physical problem is now replaced by the equivalent inclusion problem.

The total field is now obtained as the superposition of the incident field and the field induced by the presence of the mis-matches in moduli and in mass density written in terms of eigenstrains $\epsilon_{ij}^{*(1)}$ and eigenforces, π_j^*

$$\underline{\tilde{F}} = \underline{\tilde{F}}^{(i)} + \underline{\tilde{F}}^{(m)} \quad (17)$$

where $\underline{\tilde{F}}$ denotes either the displacement field u_j , the strain field ϵ_{ij} , or the stress field σ_{ij} . The superscripts (i) and (m) denote "incident" and "mis-match", respectively.

For uniform distributions of eigenstrains and eigenforces, the fields can be obtained as:

$$u_m^{(m)}(\underline{r}') = - \pi_j^* S_{jm}(\underline{r}') - C_{jkr s} \epsilon_{rs}^{*(1)} S_{jm,k'}(\underline{r}') \quad (18)$$

$$\epsilon_{mn}^{(m)} = (u_{m,n}^{(m)} + u_{n,m}^{(m)})/2 \quad (19)$$

$$\sigma_{pq}^{(m)} = C_{pqmn} \epsilon_{mn}^{(m)} \quad (20)$$

where a comma denotes partial differentiation and

$$S_{jm}(\underline{r}) = \int_{\Omega} g_{jm}(\underline{r}-\underline{r}') dV' \quad (21)$$

in which g_{jm} is the spatial part of the steady state elastic wave Green's function and Ω is the region occupied by the inhomogeneity. It is noted that the integrals S_{jm} and their derivatives must be evaluated for the region $\underline{r} > \underline{r}'$ and for the region $\underline{r} < \underline{r}'$, Ref. 12. The solution form represented in Eqs. (15-19) gives the fields inside and outside an isolated inhomogeneity of arbitrary shape.

(2) Far-field scattered quantities

Let the incident displacement field be longitudinal and of frequency ω , amplitude u_0 :

$$u_j^{(i)} = u_0 q_j \exp[i\alpha x_i k_i - i\omega t] \quad (22)$$

where $i^2 = -1$ and q_j is the unit vector in the normal direction of the plane time harmonic wave and k_i is wave vector. For a linear isotropic medium, the spatial part of the free space Green's function is well known. Substituting $g_{jm}(\underline{r}-\underline{r}')$ in Eqs. (19,21) and using the limiting concept 13:

$$\lim_{a_3 \rightarrow 0} a_3 \pi_j^* = A_j \quad \text{constants} \quad (23)$$

$$\lim_{a_3 \rightarrow 0} a_3 \epsilon_{ij}^{*(1)} = B_{ij} \quad \text{constants} \quad (24)$$

the scattered displacement $u_m^{(s)}(r,t)$ from a thin elliptical flat crack can easily be obtained as:

$$\frac{u_m^{(s)}(r,t)}{(\alpha a_1)^3 u_0} = \frac{u_m^{(m)}(r,t)}{(\alpha a_1)^3 u_0} \quad r \rightarrow \infty \quad (25)$$

$$= [(C G_m \exp i \alpha r)/\alpha r + (D H_m \exp i \beta r)/\beta r] \exp(-i \omega t)$$

where

$$G_m = - (a_2/a_1) [-\ell_m \ell_j A_j^* + (1-2\alpha^2/\beta^2) \ell_m B_{jj}^* + 2(\alpha^2/\beta^2) \ell_m \ell_k \ell_j B_{kj}^*] \quad (26)$$

$$H_m = (a_2/a_1) [-(\beta/\alpha)^3 (\ell_m \ell_j - \delta_{mj}) A_j^* - 2(\beta/\alpha)^2 \ell_k B_{km}^* + 2(\beta/\alpha)^2 \ell_m \ell_k \ell_j B_{kj}^*]$$

$$C = [j_0(\alpha r_e) + j_2(\alpha r_e)]/3 = \sin \alpha r_e / (\alpha r_e)^3 - \cos \alpha r_e / (\alpha r_e)^2 \quad (27)$$

$$D = [j_0(\beta r_e) + j_2(\beta r_e)]/3 = \sin \beta r_e / (\beta r_e)^3 - \cos \beta r_e / (\beta r_e)^2$$

$$r_e^2 = a_1^2 \ell_1^2$$

in which $m, j, k = 1, 2, 3$, and (a_1, a_2) , ℓ_m , α , β , denote the semi-axes of the flat ellipsoidal crack, direction cosines of scattered displacements, longitudinal wavenumber, shear wavenumber, respectively. Also, A_j^* and B_{jk}^* are the reduced non-dimensional form of A_j and B_{jk} , respectively, defined as follows:

$$A_j^* = A_j / (\Delta \rho \omega^2 u_0) \quad , \quad \Delta \rho = \rho' - \rho \quad , \quad (\rho' = 0 \text{ for a crack}) \quad (28)$$

$$B_{jk}^* = -B_{jk}/(i\alpha u_0) \quad (29)$$

Expressions for the differential cross section $dP(\omega)/d\Omega$ and total cross section $P(\omega)$ can be obtained as [19]:

$$\frac{dP(\omega)}{d\Omega} = \sigma^L(\theta, \phi) + (\alpha/\beta)\sigma^T(\theta, \phi) \quad (30)$$

$$P(\omega) = \int_{\Sigma} [\sigma^L(\theta, \phi) + (\alpha/\beta)\sigma^T(\theta, \phi)] d\Omega \quad (31)$$

where $d\Omega$ is the differential element of solid angle and

$$\alpha^2 \sigma^L(\theta, \phi) = (\alpha a_1)^6 [C \ G_m] [\overline{C \ G_m}] \quad (32)$$

$$\beta^2 \sigma^T(\theta, \phi) = (\alpha a_1)^6 [D \ H_m] [\overline{D \ H_m}] \quad (33)$$

in which the super bars denote complex conjugate. It is noted that the constants A_j^* and B_{jk}^* must be evaluated from the equivalence conditions Eqs. (6,7) with the use of the limiting concepts in Eqs. (22,23) and of the integration method developed in Ref. [12].

(3) Determination of A_j^* and B_{jk}^*

In Eqs. (22-26) the scattered displacement field is given in terms of the "reduced" form of the eigneforces and eigenstrains, i.e. A_j^* and B_{jk}^* . These constants must in turn be determined from the equivalence conditions. Writing the incident wave field in a Taylor series the governing simultaneous algebraic equations can be easily obtained. Since $f_{ij}[0]$ and $F_{mij}[0]$ vanish automatically, these governing equations become uncoupled and lead to a three by three system for A_j^* and a six by six system for B_{jk}^* . For a linear elastic medium, they are:

$$\Delta\rho\omega^2 u_0 f_{js} [0] A_j^* + A_s^* = - q_s \quad (34)$$

$$\begin{aligned} & \{\Delta\lambda\delta_{st} D_{mmjj} [0] + 2\Delta\mu D_{stjk} [0]\} B_{jk}^* + \{\lambda\delta_{st} B_{mm}^* + 2\mu B_{st}^*\} \\ & = - \{\Delta\lambda\delta_{st} q_m q_m + 2\Delta\mu q_s q_t\} \end{aligned} \quad (35)$$

where the subscripts $s, t, m, j, k = 1, 2, 3$ repeated subscripts denote sum from 1 to 3, and

$$4\pi\rho\omega^2 f_{js}(\underline{r}) = - \beta^2 \phi_{js} + \psi_{,mj} - \phi_{,mj} \quad (36)$$

$$\begin{aligned} 4\pi\rho\omega^2 D_{stjk}(\underline{r}) &= 2\mu[\psi_{,stjk} + \phi_{,stjk}] \\ &\quad - \mu\beta^2[\phi_{,jt}\delta_{ks} + \phi_{,js}\delta_{kt}] \\ &\quad - \lambda\alpha^2 \psi_{,mn}\delta_{jk} \end{aligned} \quad (37)$$

in which

$$\Delta\lambda = -\lambda, \Delta\mu = -\mu, \Delta\rho = -\rho$$

The ϕ - and ψ -integrals and their derivatives are evaluated by the method suggested in Ref. [12]. Retaining terms up to (αa_1) or (βa_1) of the fourth order, the constants are obtained as:

$$A_j^* = - q_j / \{u_0\omega^2 \Delta\rho f_j [0] + 1\}, \text{ no sum on } j \quad (38)$$

$$f_j [0] = (f_{11} [0], f_{22} [0], f_{33} [0]) \quad (39)$$

$$\{B_i^*\} = [b_{ij}]^{-1} [c_j] \quad i, j = 1, 2, 3 \quad (40)$$

$$\begin{aligned} \{B_{ij}^*\} &= -q_i q_j / [1 + \xi(\phi_i [0] + \phi_j [0])] , \text{ no sum on } i, j. \\ i &\neq j \\ i, j &= 1, 2, 3 \quad \xi = 1/4\pi \end{aligned} \quad (41)$$

where in Eq. (37) $B_1^* = B_{11}^*$, $B_2^* = B_{22}^*$, $B_3^* = B_{33}^*$,

$$c_j = (\Delta\lambda + 2\Delta\mu)[1 + q_j^2] , \quad j = 1, 2, 3, \quad (42a)$$

$$b_{11} = (\lambda + 2\mu) + \lambda\zeta\psi_{,jj}[0] + 2(\lambda + 2\mu)\xi\phi_{,11}[0] + 2\mu\zeta\psi_{,11}[0] \quad (42b)$$

$$b_{12} = \lambda + \lambda\zeta\psi_{,jj}[0] + 2\mu\zeta\psi_{,11}[0] + 2\xi\lambda\phi_{,22}[0] \quad (42c)$$

$$b_{13} = \lambda + \lambda\zeta\psi_{,jj}[0] + 2\mu\zeta\psi_{,11}[0] + 2\xi\lambda\phi_{,33}[0] \quad (42d)$$

$$b_{21} = \lambda + \lambda\zeta\psi_{,jj}[0] + 2\mu\zeta\psi_{,22}[0] + 2\xi\lambda\phi_{,11}[0] \quad (42e)$$

$$b_{22} = (\lambda + 2\mu) + \lambda\zeta\psi_{,jj}[0] + 2\mu\zeta\psi_{,22}[0] + 2\xi(\lambda + 2\mu)\phi_{,22}[0] \quad (42f)$$

$$b_{23} = \lambda + \lambda\zeta\psi_{,jj}[0] + 2\mu\zeta\psi_{,22}[0] + 2\xi\lambda\phi_{,33}[0] \quad (42g)$$

$$b_{31} = \lambda + \lambda\zeta\psi_{,jj}[0] + 2\mu\zeta\psi_{,33}[0] + 2\xi\lambda\phi_{,11}[0] \quad (42h)$$

$$b_{32} = \lambda + \lambda \zeta \psi_{,jj}[0] + 2\mu \zeta \psi_{,33}[0] + 2\lambda \xi \phi_{,22}[0] \quad (42i)$$

$$b_{33} = (\lambda + 2\mu) + \lambda \zeta \psi_{,jj}[0] + 2\mu \zeta \psi_{,33}[0] + 2(\lambda - 2\mu) \xi \phi_{,33}[0] \quad (42j)$$

Note that $b_{ij} \neq b_{ji}$. In Eq. (38), $\phi_1 = \phi_{11}^{[0]}$, $\phi_2 = \phi_{22}[0]$, $\phi_3 = \phi_{33}[0]$.

The f -, ϕ - functions are given as:

$$4\pi\rho\omega^2 f_{js}[0] = -\beta^2 \phi[0] \delta_{js} + \psi_{,js}[0] - \phi_{,js}[0] \quad (43)$$

$$\beta^2 \phi[0] = \pi a_1 a_2 \beta^2 \{I_0 - ((\beta a_1)^2/16)I_1 + i(8/3)\beta\} \quad (44)$$

$$\phi_{,11}[0] = -[(\pi a_1^3 a_2 \beta^4)/12] \cdot I_1 \quad (45)$$

$$\phi_{,22}[0] = -[(\pi a_1 a_2^3 \beta^4)/12] \cdot I_1 \quad (46)$$

$$\phi_{,33}[0] = 0 \quad (47)$$

in which

$$I_0 = \int_0^\infty \frac{d\psi}{\Delta(\psi)} = F(\theta, k) \cdot \frac{2}{a_1} \quad (48)$$

$$I_1 = \int_0^\infty \frac{\psi d\psi}{(a_1^2 + \psi)\Delta(\psi)} = \frac{2}{a_1} \left\{ \frac{E(\theta, k)}{k^2} - \frac{k'^2}{k^2} F(\theta, k) \right\} \quad (49)$$

$$I_2 = \int_0^\infty \frac{\psi d\psi}{(a_2^2 + \psi)\Delta(\psi)} = \frac{2}{a_1} \left\{ \frac{F(\theta, k)}{k^2} - \frac{E(\theta, k)}{k^2} \right\} \quad (50)$$

$$F = \int_0^\theta \frac{d\omega}{(1-k^2 \sin^2 \omega)^{1/2}}, \quad E = \int_0^\theta (1-k^2 \sin^2 \omega)^{1/2} d\omega \quad (51)$$

and as $a_3 \rightarrow 0$, $\theta \rightarrow \pi/2$, $k^2 \rightarrow (1-a_2^2/a_1^2)$ and $k'^2 = (1-k^2) \rightarrow a_2^2/a_1^2$, if $a_1 > a_2$. If $a_3 \rightarrow 0$ and $a_1 = a_2$, we have $I_0 = \pi/a_1$, $I_1 = I_2 = \pi/2a_1$. The 4ψ - functions are obtained by replacing β with α in the ϕ - functions.

(4) Stress Intensity Factor (SIF) and Crack Opening Displacement (COD) - preliminary results.

The general expression for the total displacement field can be easily obtained, by using Eq. (2) and the expression for the mis-match displacement field [9], as follows:

$$u_m(\underline{r}) = f_{mj}(\underline{r})A_j + f_{mjk}(\underline{r})A_{jk} + \\ + F_{mij}(\underline{r})B_{ij} + F_{mijk}(\underline{r})B_{ijk} + \dots \quad (52)$$

To find the crack face opening displacement the functions $f^<$ and $F^<$ must be evaluated for \underline{r} inside the crack.

In this section, the solution for a crack is that degenerated from an ellipsoidal void by letting $a_3 \rightarrow 0$. The crack opening displacement for an elliptical crack can be defined as

$$\Delta u_i = 2u_i = h(\epsilon_{3i}^* + \pi_i^*) \quad (53)$$

where h is the half thickness of the ellipsoidal void in the x_3 -direction:

$$h = a_3(1 - x_1^2/a_1^2 - x_2^2/a_2^2)^{1/2} \quad (54)$$

For the case of uniformly distributed eigenstrains and eigenforces, the crack opening displacement is, therefore, obtained as:

$$\Delta u_j = [(A_i + B_{3j})a_3h]_{a_3 \rightarrow 0}$$

or
$$\frac{\Delta u_j}{u_0} = \left(1 - \frac{x_1^2}{a_1^2} - \frac{x_2^2}{a_2^2}\right)^{1/2} \left[\frac{\Delta \rho}{\rho} A_i^* + B_{3i}^* \right] \quad (55)$$

where u_0 is the amplitude of the incident wave field and A_j and B_{3j} are given by Eqs. (38-41).

The stress intensity factors can be easily obtained by finding the strain energy change W , i.e. the interaction energy, due to the presence of the crack and then differentiate with respect to the crack dimension. An approximation for low frequency range can be obtained by using the SIF-COD relation [11]. Surface-wave interaction with an edge crack was studied in [33].

(5) Numerical calculations and graphical displays

It is clear from Eqs. (38-41) that the uniformly distributed eigenstrains and eigenforces, in their reduced form, B_{ij} and A_j , respectively, are dependent only upon the incident wave field characteristics and the geometric factors of the inhomogeneity for a given material system. It is observed from Eqs. (38-42) that B_i and B_{ij} as functions of dimensionless wavenumber of a_1 or a_2 would exhibit large peak values at certain incident wave frequencies when $b_{ij} \rightarrow 0$. The values of these critical frequencies depend only upon the matrix elastic moduli, the crack dimensions and the measurement direction. For a given crack aspect ratio, a_2/a_1 , and a measurement direction, the difference in frequencies at subsequent peak values is proportional to a_1 , the largest dimension of the crack, Fig. 9.

Computational data of elastic wave scattering due to a flat crack embedded in any given isotropic material system can be obtained by employing Eqs. (25-29) and (34-51) with $\Delta\rho = -\rho$, $\Delta\lambda = -\lambda$ and $\Delta\mu = -\mu$. Scattered displacement amplitudes can easily be obtained for any given crack aspect ratio.

(a) Elliptical Crack: In Figs. 7-9 computational data for back scattered situation are displayed for a crack in aluminum of different crack aspect ratios, i.e. $a_2/a_1 = 1, 1/2, 1/10$ at various wave incident angle. Critical frequencies are observed to be present in B_{ij}^* but no critical frequencies are identified for A_j^* , $j = 1, 2, 3$ Figs. 10-13. The critical frequencies for $C|G_m|$ and $D|H_m|$ are, however, clearly identified. The position along the αa_1 axis at which the first critical frequency occurs depend upon the crack aspect ratio and matrix elastic moduli. Sufficiently small increment in αa_1 must be used in order not to miss any peak values. Since the solution form given in Eqs. (25-27, 38-51) is analytic in frequency, this can easily be achieved. It is noted that most of the scattered energy is carried by the transverse components of the scattered displacement, i.e. $|DH_m| > |CG_m|$, Figs. 14, 15.

(b) Penny-shaped Crack: In Fig. (7), the reduced longitudinal back scattered displacements at $\theta = \phi = 45^\circ$ and the same crack aspect ratio, $a_2/a_1 = 1$, are shown for a crack in structural steel. The change in the absolute value of the crack size causes a compaction of the peaks in $|CG_m|$ along the axis of dimensionless wavenumber αa_1 . It is of interest to observe that the difference in frequencies at subsequent peak values, e.g. the first and the second peaks, Δf_1 , is inversely proportional to a certain power of a_1 . This fact can easily be algebraically demonstrated by using Eq. (42) to find the critical frequencies, i.e. finding the roots of the following algebraic equation:

$$C_1 \omega^4 + C_2 \omega^2 = C_3 \quad (56)$$

where the constants C_1 , C_2 and C_3 are obtained by directly substituting Eqs. (42-51) into Eqs. (38,41) for each component of b_{ij} . Since

$$\Delta f = \Delta(\alpha a_1) v_L / a_1 \quad (57)$$

a scale in frequency is also given in Figs. (10-20). It is clearly seen from these figures that for small cracks, accuracy of solution at lower dimensionless wavenumber is advisable and for larger cracks a wider validity along the axis of dimensionless wavenumber is of interest. It should be noted that the peaks in these figures are very large.

In Figs. 14, 19 and 20, computational data for back-scattered reduced transverse displacement amplitude are displayed for different aspect ratios, $a_2/a_1 = 0.01$ and $a_2/a_1 = 0.50$. For a fixed aspect ratio, again, it is clear that the difference in frequencies at subsequent peaks, say Δf_1 or f_1 , is inversely proportional to the third power of a_1 , the largest dimension of the crack. All calculations are done for given αa_1 . Note that the absolute scales in frequency are quite different for different a_1 with same aspect ratio.

Dynamic Moduli and Damage of Composites

Consider the problem of the inhomogeneous media, as illustrated in Fig. 4, under a plane time-harmonic indent wave field, Eq. (22). The true composite thus occupies the whole region and possesses effective moduli ζ^* and mass density ρ^* . To determine the effective moduli and mass density, an average theorem over strain energy and kinetic energy

is used. The effective properties are found to depend upon a fourth rank tensor \underline{A} and a second rank tensor \underline{D} . A self-consistent scheme is then developed for the determination of these tensors.

(1) Average theorem

To determine the effective moduli and mass density, the following definitions are used:

$$\langle \underline{\sigma} \rangle = \underline{C}^* \langle \underline{\epsilon} \rangle \quad (58)$$

$$\langle \underline{\sigma} \underline{\epsilon} \rangle = \underline{C}^* \langle \underline{\epsilon} \rangle \langle \underline{\epsilon} \rangle \quad (59)$$

$$\langle \rho \underline{v} \rangle = \rho^* \langle \underline{v} \rangle \quad (60)$$

$$\langle \rho \underline{v} \cdot \underline{v} \rangle = \rho^* \langle \underline{v} \rangle \langle \underline{v} \rangle \quad (61)$$

where $\underline{\sigma}$, $\underline{\epsilon}$, \underline{v} are the stress, strain and velocity fields, the underlined ~ denotes a tensorial quantity, and the angled brackets $\langle \rangle$ denote the volume average of a field quantity, Eq. (10). The left-hand-side and the right-hand-side in Eqs. (59) and (61) can be shown to be equivalent under the so-called Hill's condition [20], $\langle \underline{\sigma} \rangle \langle \underline{\epsilon} \rangle = \langle \underline{\sigma} \underline{\epsilon} \rangle$. From Eqs. (60) and (61) it is clear that the kinetic energy per unit volume of the effective medium can be made equal to that of the physical medium if and only if a frequency dependent mass density is defined. It can also be seen, that this is the same as requiring by again using the Hill's conditions $\langle \rho, \underline{v} \cdot \underline{v} \rangle = \langle \rho \underline{v} \rangle \langle \underline{v} \rangle$ the average linear momentum per unit volume be the same as the effective linear momentum per unit volume, Eq. (60). These conditions, Eqs. (60) and (61) are not met if the static definition of effective mass density is used.

Let f_r denote the volume fraction of the (r)th inclusion material, then the volume average of the stress and velocity fields $\underline{\sigma}$ and $\underline{\dot{u}}$, respectively, are

$$\langle \underline{g} \rangle = \sum_{r=0}^n f_r \underline{g}^{(r)} \quad (62)$$

$$\langle \underline{\dot{u}} \rangle = \sum_{r=0}^n f_r \underline{\dot{u}}^{(r)} \quad (63)$$

where $\underline{g}^{(r)} = \underline{C}^{(r)} \underline{\varepsilon}^{(r)}$, $(r) = 0, 1, 2, \dots, n$,
and for time-harmonic situation

$$\underline{\dot{u}} = \underline{v} = -i\omega \underline{u} \quad (64)$$

Equations (59) and (61) can be rearranged, by using Eqs. (62) and (63), as

$$\langle \underline{\sigma \varepsilon} \rangle = C^0 \langle \underline{\varepsilon} \rangle \langle \underline{\varepsilon} \rangle + \sum_{r=1}^n f_r \underline{C}^{(r)} \underline{\varepsilon}^{(r)} \langle \underline{\varepsilon} \rangle \quad (65)$$

$$\langle \rho \underline{v} \cdot \underline{v} \rangle = \rho^0 \langle \underline{v} \rangle \langle \underline{v} \rangle + \sum_{r=1}^n f_r \Delta \rho^{(r)} \underline{v}^{(r)} \langle \underline{v} \rangle \quad (66)$$

Consider now the case when the solution form possesses a linear relation between the velocity and strain fields in the (r) th component and the average velocity and strain fields of the effective medium, i.e.

$$\underline{\varepsilon}^{(r)} = \underline{A}^{(r)} \langle \underline{\varepsilon} \rangle \quad (67)$$

$$\underline{v}^{(r)} = \underline{D}^{(r)} \langle \underline{v} \rangle \quad (68)$$

where $\underline{A}^{(r)}$ and $\underline{D}^{(r)}$ are tensors of fourth and second rank that must be determined with a suitable scheme. The substitution of Eqs. (67) and (68) in Eqs. (65) and (66) leads to

$$\langle \underline{\sigma} \underline{\epsilon} \rangle = \{ \underline{C}^0 + \sum_{r=1}^n f_r \Delta \underline{C}^{(r)} \underline{A}^{(r)} \} \langle \underline{\epsilon} \rangle \langle \underline{\epsilon} \rangle \quad (69)$$

$$\langle \rho \underline{v} \underline{v} \rangle = \{ \rho^0 + \sum_{r=1}^n f_r \Delta \rho^{(r)} \underline{D}^{(r)} \} \langle \underline{v} \rangle \langle \underline{v} \rangle \quad (70)$$

A comparison of the above with Eqs. (59) and (61), the expressions for the dynamic effective moduli and mass density are obtained as follows:

$$\underline{C}^* = \underline{C}^0 + \sum_{r=1}^n f_r \Delta \underline{C}^{(r)} \underline{A}^{(r)} \quad (71)$$

$$\rho^* = \rho^0 + \sum_{r=1}^n f_r \Delta \rho^{(r)} \underline{D}^{(r)} \quad \underline{D}^{(r)} = \underline{D}_{jj}^{(r)} / 3 \quad (72)$$

It is important to note that the tensors $\underline{A}^{(r)}$ and $\underline{D}^{(r)}$ are frequency dependent and they replace the static expressions when frequency approaches zero and when proper cares are taken. The assumption of a general linear dependence between $\underline{v}^{(r)}$ and $\langle \underline{v} \rangle$, Eq. (68), will have to be specialized such that the second rank tensor \underline{D} will degenerate into a scalar. This specialization is automatic for randomly distributed spheres where $D_{mj} = D \delta_{mj}$, in which $D = D_{11} = D_{22} = D_{33}$.

To determine the explicit form of the tensors $\underline{A}^{(r)}$ and $\underline{D}^{(r)}$, the strain and velocity fields in the (r) th component are determined by using the method of equivalent inclusion as presented in the previous section.

- (2) A self consistent scheme for the determination of the effective properties

The tensor fields $\underline{A}^{(r)}$ and $\underline{D}^{(r)}$ are of ranks four and two, respectively, and they are functions of wavenumbers, geometric properties, and effective and inclusion material properties. Let the average strain and velocity be the same as those derived from the incident wave field, then the governing conditions for determining $\underline{A}^{(r)}$ and $\underline{D}^{(r)}$ for the (r) th inclusion are simply obtained by rewriting the equivalence conditions for an effective medium with mass density ρ^* and moduli C^* :

$$\underline{C}^{(r)}(\underline{\varepsilon}^{(i)} + \underline{\varepsilon}^{(m)}) = \underline{C}^*(\underline{\varepsilon}^{(i)} + \underline{\varepsilon}^{(m)} - \underline{\varepsilon}^*) , \text{ in } \Omega_r \quad (73)$$

$$\rho^{(r)} \omega^2 (\underline{u}^{(i)} + \underline{u}^{(m)}) = \rho^* \omega^2 (\underline{u}^{(i)} + \underline{u}^{(m)}) - \underline{\pi} , \text{ in } \Omega_r \quad (74)$$

where the superscripts (i) and (m) stand for "incident" and "mis-match", respectively. Slightly different approaches for finding effective moduli for heterogeneous material are given in [20,23] that apply to static cases with different constituents and situations. The strength and fracture responses of composites have been investigated [17,18, 20-23].

- (3) Effective properties of two-component media: randomly distributed spheres

Let the moduli, mass density for the matrix, inclusion material particles and the effective medium be denoted by C_{jkr}^0 , ρ^0 ; C_{jkr}' , ρ' ; and C_{jkr}^* and ρ^* , respectively. Using the elastodynamic solution for a single ellipsoidal inhomogeneity, the displacement and strain fields inside an inhomogeneity are found, when the average displacements are made equal to the incident time-harmonic plane wave field, to be:

$$[\underline{u}^{(m)}(\underline{r})] = \underline{D}(\underline{r})\underline{u}^{(i)} - \underline{\delta}(\underline{r})\underline{u}^{(i)}, \text{ in } \Omega_r \quad (75)$$

$$[\underline{\varepsilon}^{(m)}(\underline{r})] = \underline{A}(\underline{r})\underline{\varepsilon}^{(i)} - \underline{\delta}(\underline{r})\underline{u}^{(i)}, \text{ in } \Omega_r \quad (76)$$

in which the superscripts (m) and (i) denote "mis-match" and "incident", respectively. Employing the volume averaging process as described by Eq. (10) and substituting in Eqs. (65,66), the effective properties are easily defined as follows:

$$\rho^* = \rho + f\Delta\rho D, \quad D = \frac{1}{3}D_{jj} \quad (77)$$

$$\underline{C}^* = \underline{C} + f\Delta\underline{C}\underline{A} \quad (78)$$

where f is volume fraction of inclusion material. The tensor fields \underline{D} and \underline{A} are:

$$D_{mj} = -\langle f_{mj}(\underline{r}) \rangle / \{f_{MJ}[0] + 4\pi(\rho' - \rho^*)\omega^2\}, \text{ no sum on } M, J. \quad (79)$$

$$A_{mnpq} = \langle F_{mjk,n}(\underline{r}) + F_{njk,m}(\underline{r}) \rangle S_{jkpq} / 2\rho^*\omega^2 \quad (80)$$

in which the tensors \underline{f} and \underline{F} are defined as:

$$4\pi\rho\omega^2 f_{mj}(\underline{r}) = -\beta^2 \phi_{mj} + \psi_{,mj} - \phi_{,mj} \quad (81)$$

$$4\pi\rho\omega^2 F_{mij}(\underline{r}) = -[\lambda\alpha^2 \psi_{,m} \delta_{ij} + 2\mu\beta^2 \phi_{,i} \delta_{mj} - 2\mu\psi_{,mj} + 2\mu\phi_{,mj}] \quad (82)$$

The ϕ - and ψ -integrals given in Eqs. (15,16), etc. are the volume integrals associated with the inhomogeneous Helmholtz equation. They can be carried out for an ellipsoidal region by expanding $(\exp ikR)/R$ in

Taylor series expansions with respect to r' , for $r > r'$ and with respect to r for $r < r'$. Here k can be either α or β . Details are given in Ref. [12]. This type of expansion for the integrand is particularly useful in determining the coefficients of a "polynomial" distribution of π_j^* and $\epsilon_{ij}^{*(1)}$.

The fourth rank tensor $S_{j k p q}$ is the connecting tensor between the eigenstrains and the applied strains, i.e.

$$\epsilon_{jk}^{*(1)} = S_{j k p q}^{(a)} \rho_{pq}^*, \quad S_{j k p q} = S_{j k p q}(\underline{C}, a_i, \omega) \quad (83)$$

for the case of uniform eigenstrains and eigenforces. In developing these expressions the volume average of the ϕ -integrals, must be evaluated. Finally, it should be noted that ρ^* and \underline{C}^* are complex where the real and imaginary parts are associated with the velocity and attenuation, respectively.

(4) Example: spherical inclusion materials

Let the spherical inclusion materials of radius "a" be randomly distributed over the whole volume of the matrix. If the matrix and the inhomogeneities are isotropy, the effective medium is also isotropic. It is straight forward to show that

$$\begin{aligned} D_{mj} &= \delta_{mj} \{ - \langle f_{33}(\underline{r}) \rangle / f_{33}[0] + 4\pi(\rho' - \rho^*)\omega^2 \} \\ &= \delta_{mj} D \end{aligned} \quad (84)$$

and

$$\begin{aligned} S_{j k p q} &= S_{k j p q} = S_{j k q p} = S_{p q j k} \\ S_{111} &= S_{222} = S_{333} = C_1 \\ S_{2323} &= S_{1313} = S_{1212} = C_3 \\ S_{1122} &= S_{1133} = S_{2233} = C_2 \end{aligned} \quad (85)$$

where

$$C_1 = (C_1^* + C_2^* - 2GC_2^*)/[C_1^*]^2 + C_1^*C_2^* - 2(C_2^*)^2]$$

$$C_2 = (C_1^*G - C_2^*)/[C_1^*]^2 + C_1^*C_2^* - 2(C_2^*)^2]$$

$$C_3 = \{2F_{122,1}[0] + \mu^*/(\mu' - \mu^*)\}$$

$$C_2^* = GF_{111,1}[0] + (G+1)F_{122,1}[0] + H$$

$$C_1^* = F_{111,1}[0] + 2GF_{122,1}[0] - F$$

$$F \equiv -(\lambda^* + 2\mu^*)/G$$

$$G = (\lambda' - \lambda^*)/[(\lambda' - \lambda^*) + 2(\mu' - \mu^*)]$$

$$H = \lambda^*/G$$

Following the theory developed in the previous sections, the effective moduli and mass density are found to be

$$\rho^* = \rho + f\Delta\rho D \quad (86)$$

$$\lambda^* = \lambda + f[\Delta\lambda(A_{1111} + 2A_{1122}) + 2\Delta\mu A_{1122}] \quad (87)$$

$$\mu^* = \mu + f\Delta\mu(A_{1212} + A_{1221}) \quad (88)$$

$$K^* = K + f[(A_{1111} + 2A_{1122})\Delta\lambda + (2/3[3A_{1122} + A_{1212} + A_{1221}]\Delta\mu)] \quad (89)$$

It is clearly seen that the velocities are dispersive. At frequency range above that of the Rayleigh limit, this phenomenon is pronounced. From Figs. 21-23, the bulk moduli, shear moduli and longitudinal velocities are shown as functions of volume concentration of spherical inclusion materials for the cases of aluminum spheres in germanium, for different dimensionless wavenumbers αa . For a given fixed concentration, the moduli K^* , μ^* and velocity V_L and V_T are increased as the dimensionless wavenumber αa is increased. The dispersiveness of effective shear modulus is minimal and that of effective bulk modulus is more pronounced, Figs. 24,25.

As an example of application to detect localized damage by void nucleation, let all small voids be locally nucleated within a localized small region Ω of radius R , Fig. 26. The effective moduli of this composite can therefore be obtained from Eqs. (71,72). If void nucleation outside the region Ω can be ignored, then the scattering of the composite sphere can easily be obtained. Using the computer program developed in [24], the scattering cross section for a composite sphere consisted of small voids in titanium is displayed as a function of dimensionless wavenumber for different concentration of voids, Fig. 27. It is noted that as the volume fraction of voids inside Ω is changed, the effective properties, ρ^* , λ^* and μ^* are also changed. Hence the attenuation effect is pronounced as the concentration of voids is increased. The scattering cross section, which is essentially proportional to the attenuation [19], increases with increasing concentration f_v . It appears that these curves can be used to locate and calibrate porosity in a structural component. Dynamic effective properties for this material system are presented in [15].

CLOSING REMARKS

The mechanics aspects of the characterization of microfracture and microdamage by ultrasonics are studied by first looking into the scatter of elastic waves by a flat ellipsoidal crack and then by seeking an average measure of damage. The work was a part of a three-year program.

The solution to the direct scattering of a flat ellipsoidal crack is presented by using the extended version of Eshelby's method of equivalent inclusion and a limiting concept. The solution is thus obtained by collapsing an ellipsoidal void to a flat crack, say taking $a_3 \rightarrow 0$. The orientation of the crack is assumed to be known. The solution form is analytic in incident wave frequency and is in terms of the eigenstrains and eigenforces which are governed by the incident wave characteristics and the equivalence conditions, Eqs. (6,7). The solution agrees with the Rayleigh limit [8] and goes beyond it. The solution appears to possess a range of validity along the axis of dimensionless wavenumber αa_1 less than 2π .

There are identifiable critical frequencies at which the scattered displacement amplitudes become infinite in value. For any given aspect ratio of the crack axes, the difference in critical frequencies at subsequent peaks is inversely proportional to the crack size. A procedure for ultrasonic crack sizing is thus suggested and described as follows.

First, it is assumed that the orientation of the crack plane is known or can be determined by finding the direction of maximum scattered energy. The differences in frequencies at peak values in the frequency spectra at different look angles can then be used to determine the aspect ratio and the crack size. The details of the inverse problem of crack sizing should be a research program by itself,

Other areas of research that should be done and can be done are the determination of the on-set of microcracking due to orientation and geometry by using the crack opening displacement and stress intensity factor. Since these quantities can be written in terms of the eigenstrains and eigenforces, they can easily be related to the scattered

displacements. Earlier references on microcracking in ceramics can be found in [16,17,32,37]. Detection and determination of subsurface cracks are also of substantial interest in the non-destructive testing (NDT) aspect of the science and technology of fracture. The use of the concepts in Section III.2 for possible characterization of transducer response is also of interest [33].

The velocity and attenuation of ultrasonic waves in two-phase media are studied by using a self-consistent averaging scheme. It is required that the effective medium to possess the same strain and kinetic energy as the physical medium. The concept of volume averaging for physical quantities is employed and the solution depend upon the scattering of a single inhomogeneity. The theory is general in nature and can be applied to multi-component material system. Since the scattering of an ellipsoidal inhomogeneity is known, the average theorem presented in this report can be used to study the velocity and attenuation of distributed inhomogeneities of shapes such as disks, short fibres, etc. The introduction of the orientation of these inhomogeneities besides their sizes as in the spherical geometry will necessarily induce anisotropy in the effective medium. Fracture toughness and localized damage can be studied [4].

Results for randomly distributed spherical inclusions of radius "a" are presented. Effective moduli and mass density are found to be dispersive. The case of a simple model of localized damage is studied. Since it is well known that porosity is directly related to the strength of rocks and ceramics it appears that the theoretical study of velocity and attenuation in two-phase media may be a viable means for data analysis in ultrasonic evaluation of dynamic material properties[†] for composite bodies [23,36]. Manufacturing processing, such as rolling, sheet metal forming, drawing, etc. often involves plastic flow and fracture in the material. Porosity and/or plastic strains induced

[†]These are defined as material properties that are obtained by using ultrasonics.

or contained in the material introduce residual stresses and anisotropy in the material and thereby limit the amount of deformation to fracture with a directional dependence. Continuous monitor of (1) current global moduli, strength and fracture toughness and (2) localized damage such as necking may be of importance in the design and optimization of manufacturing procedures. One convenient means for such continuous monitor is via ultrasonic velocity and attenuation methods. If effective moduli and associated phase velocity and attenuation are determined for identifiable damage parameters, then the information can be used to reconstruct the size and shape of an internal damage zone. Together with well developed damage theory, correlation relations with sound theoretical basis such as that described in [3,4] will lead to prediction of failure or optimum design of processes.

Other possible application may include soil-structure or fluid-structure interaction problems [34,35] where a combination of analysis and numerical approach may be involved. The development to cover large strain formulation may be needed.

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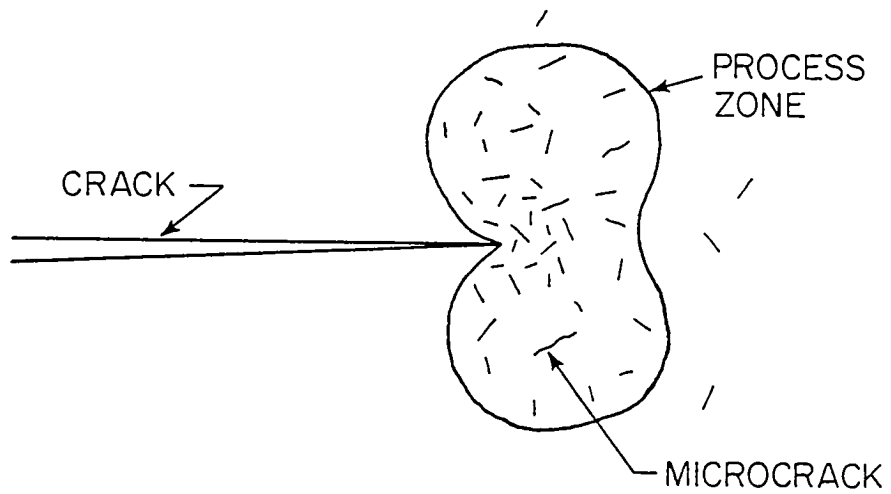


Fig. 1 Microcrack process zone within the vicinity of the crack tip.

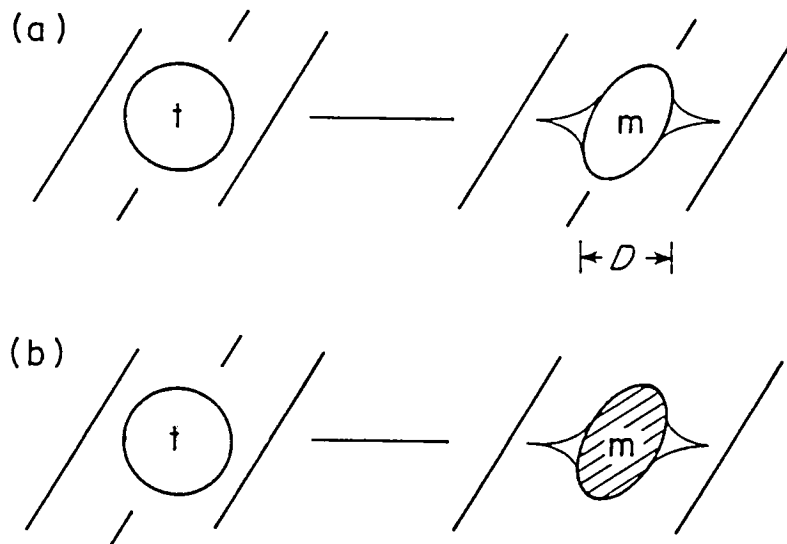


Fig. 2 (a) Transformation + microcracking,
 (b) transformation + microcracking
 + twinning. (From F. Lange, 1982)

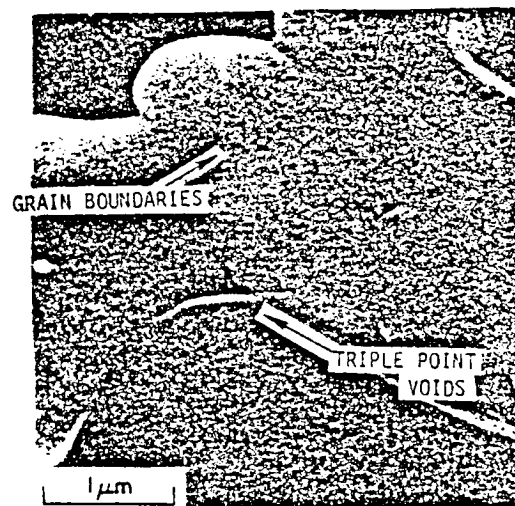


Fig. 3 A scanning electron micrograph of a typical triple point void in sintered Al_2O_3 . (From A.G. Evans.)

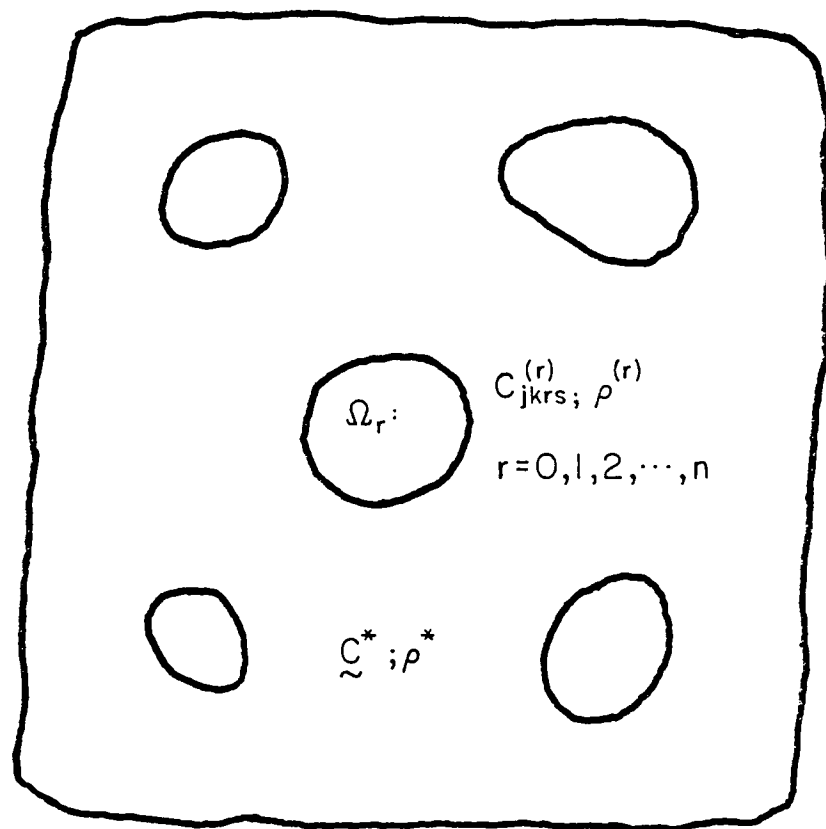


Fig. 4 Inhomogeneous media with multi-components.

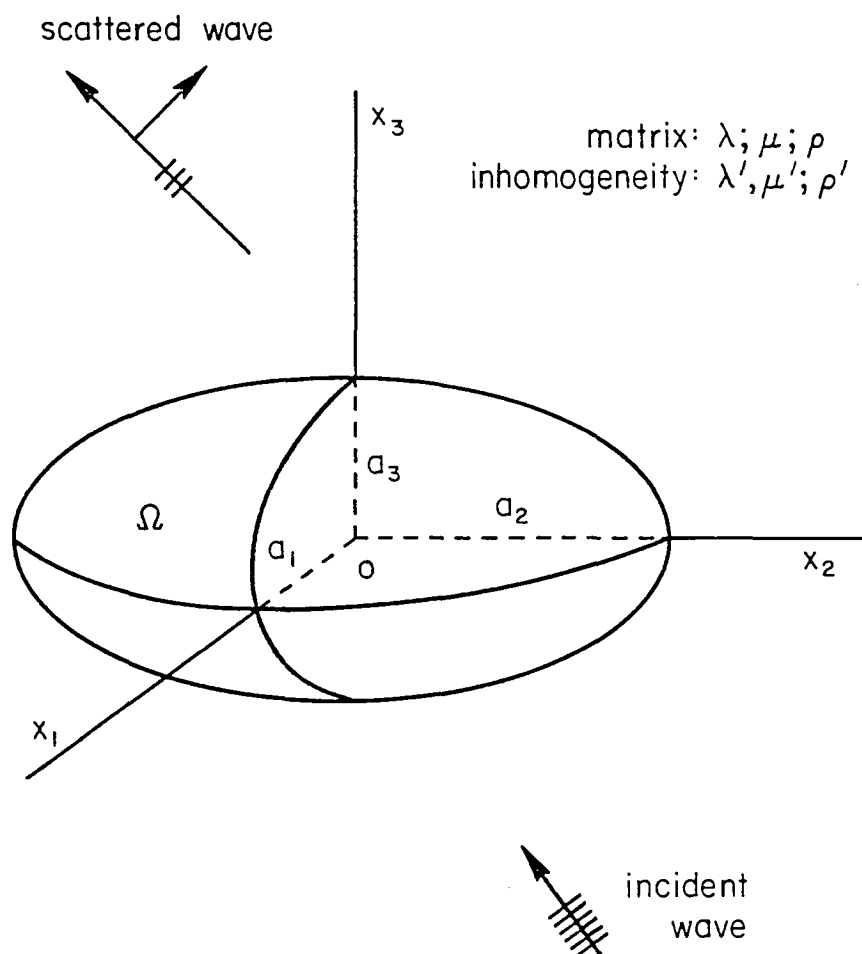


Fig. 5 An ellipsoidal inhomogeneity embedded in elastic matrix

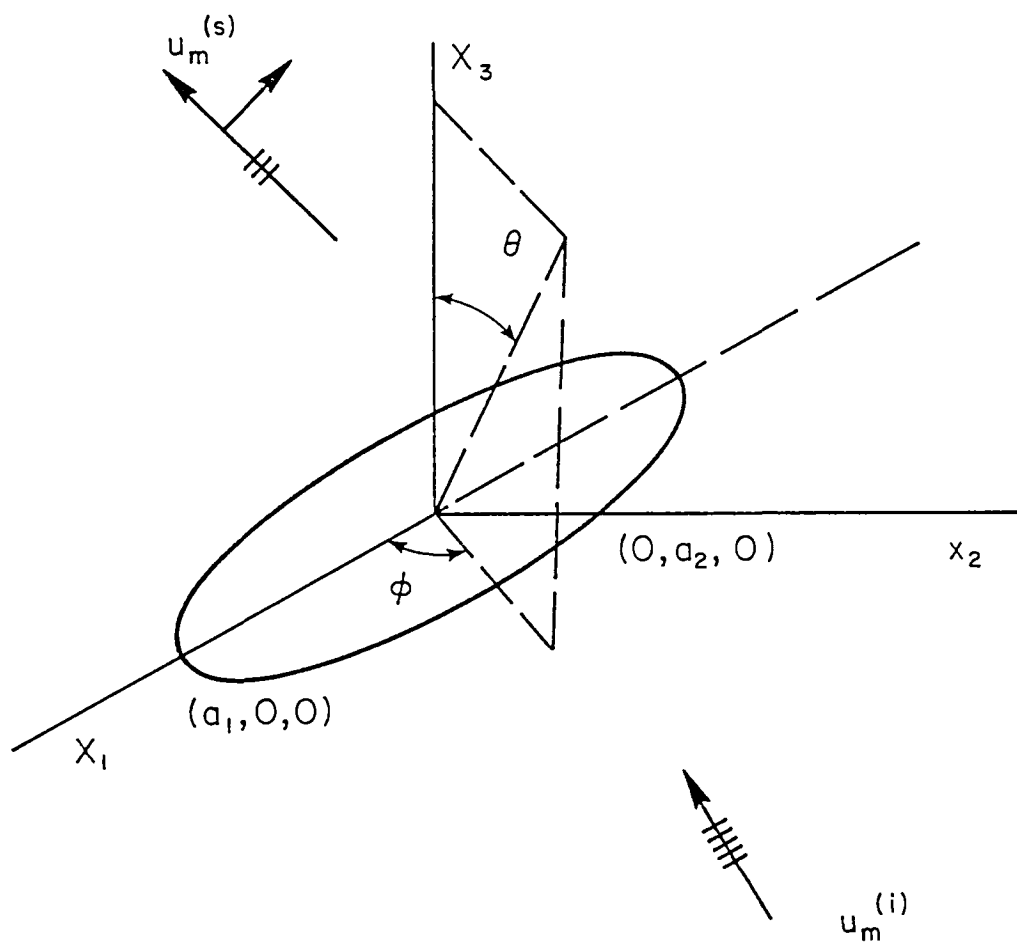


Fig. 6 An elliptical crack: $a_3 \rightarrow 0$.

CRACK IN ALUMINUM

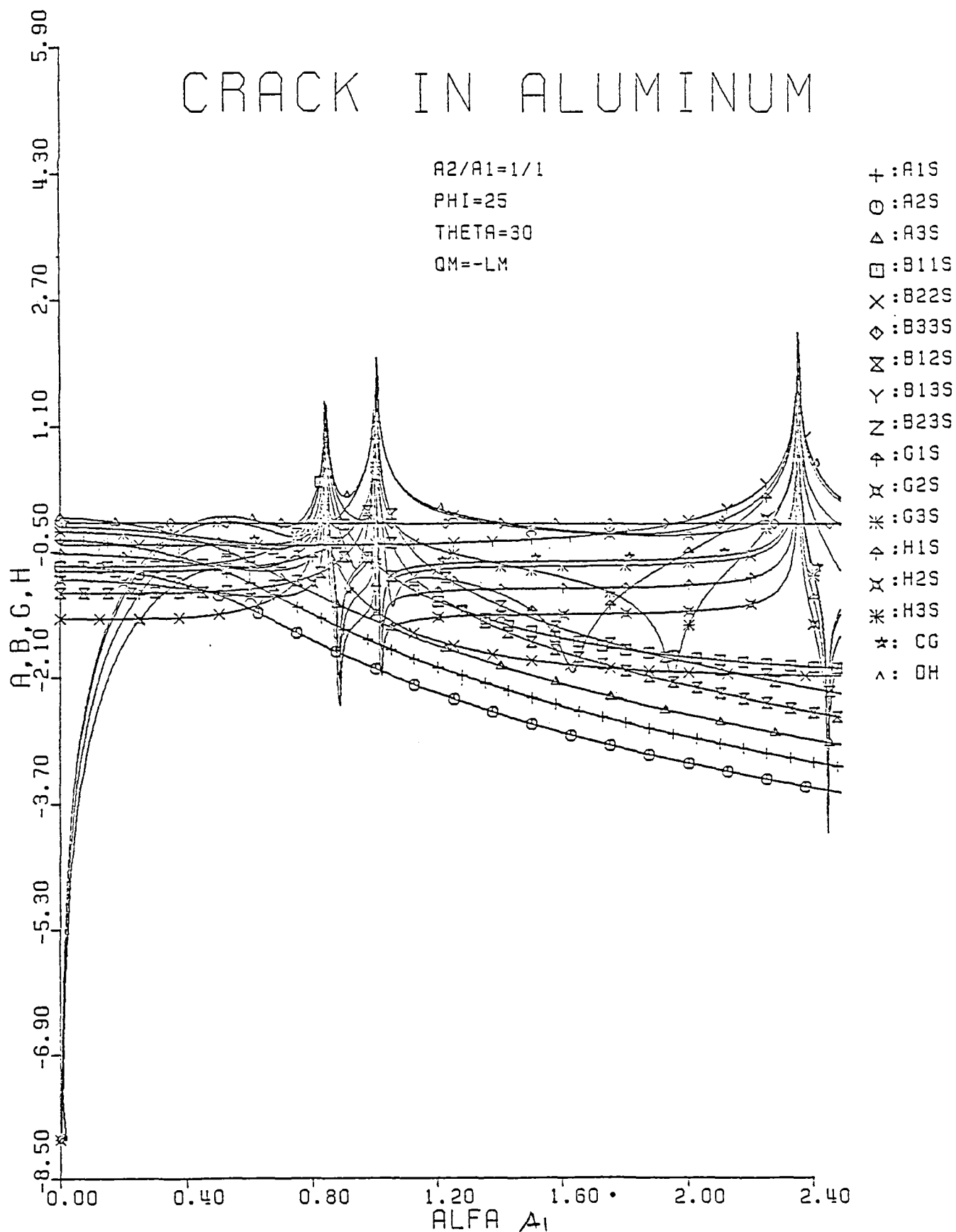


Fig. 7 Back scattered data.

VOID CRACK IN AL

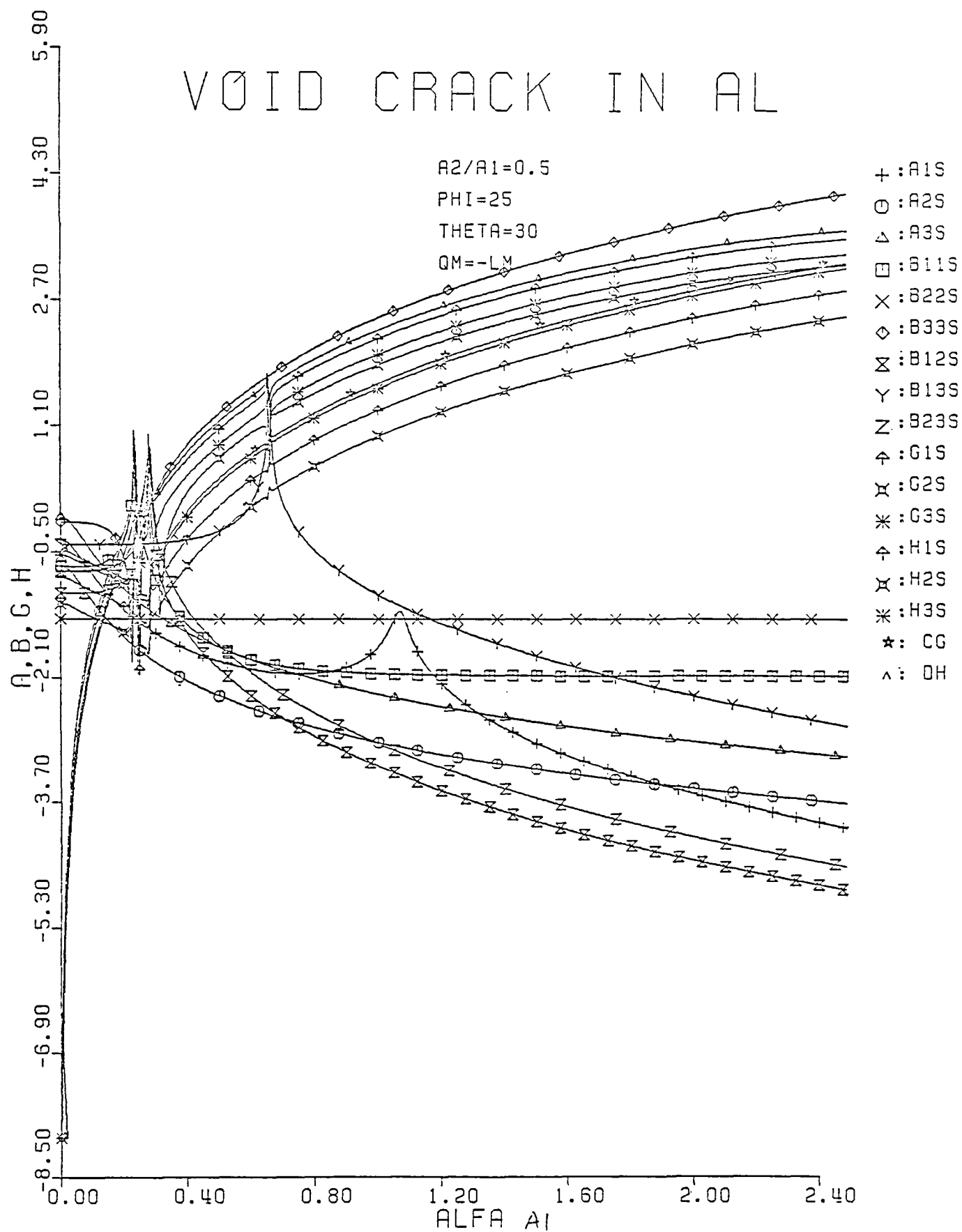


Fig. 8 Back scattered data.

CRACK IN ALUMINUM

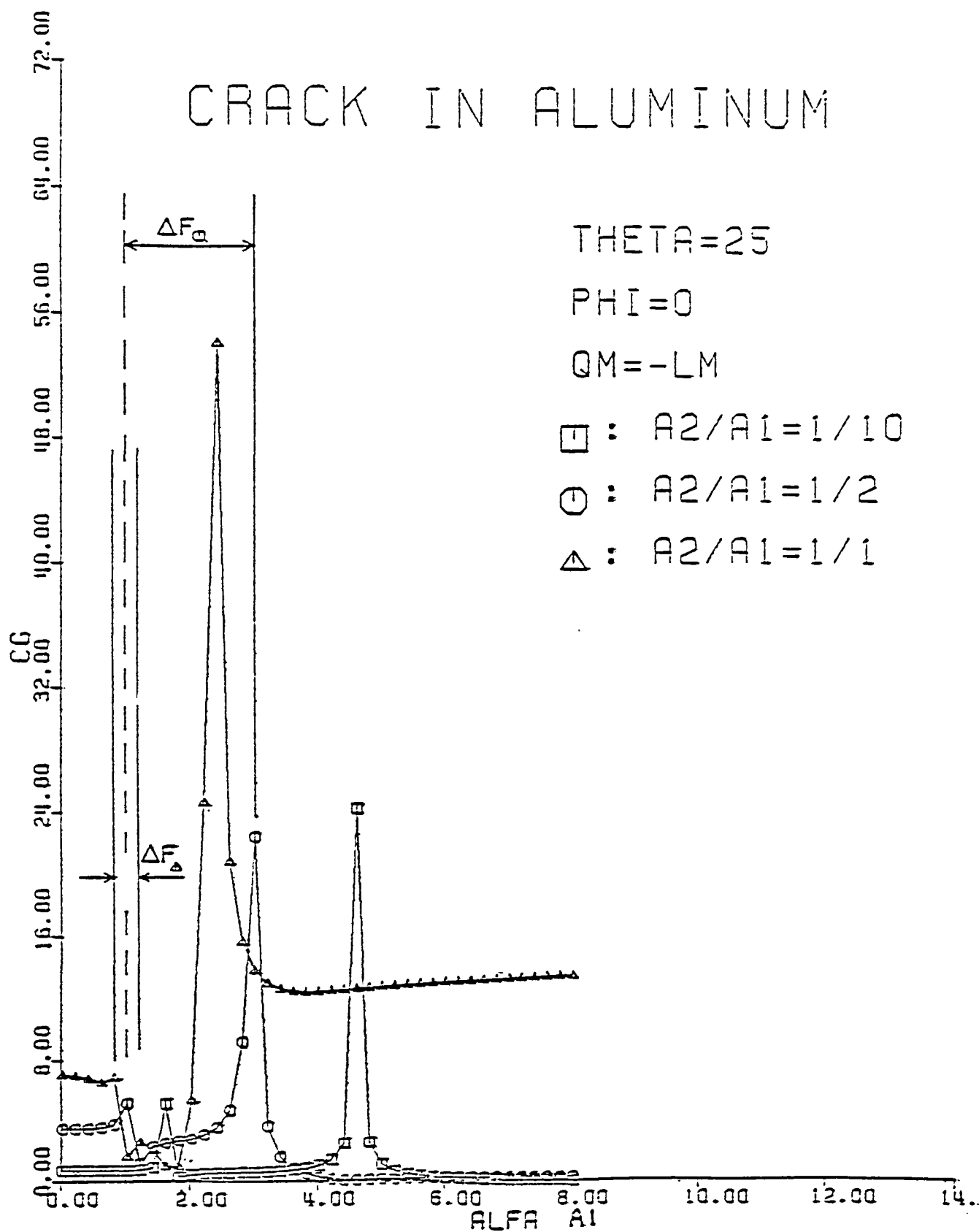


Fig. 9 Back scattered data.

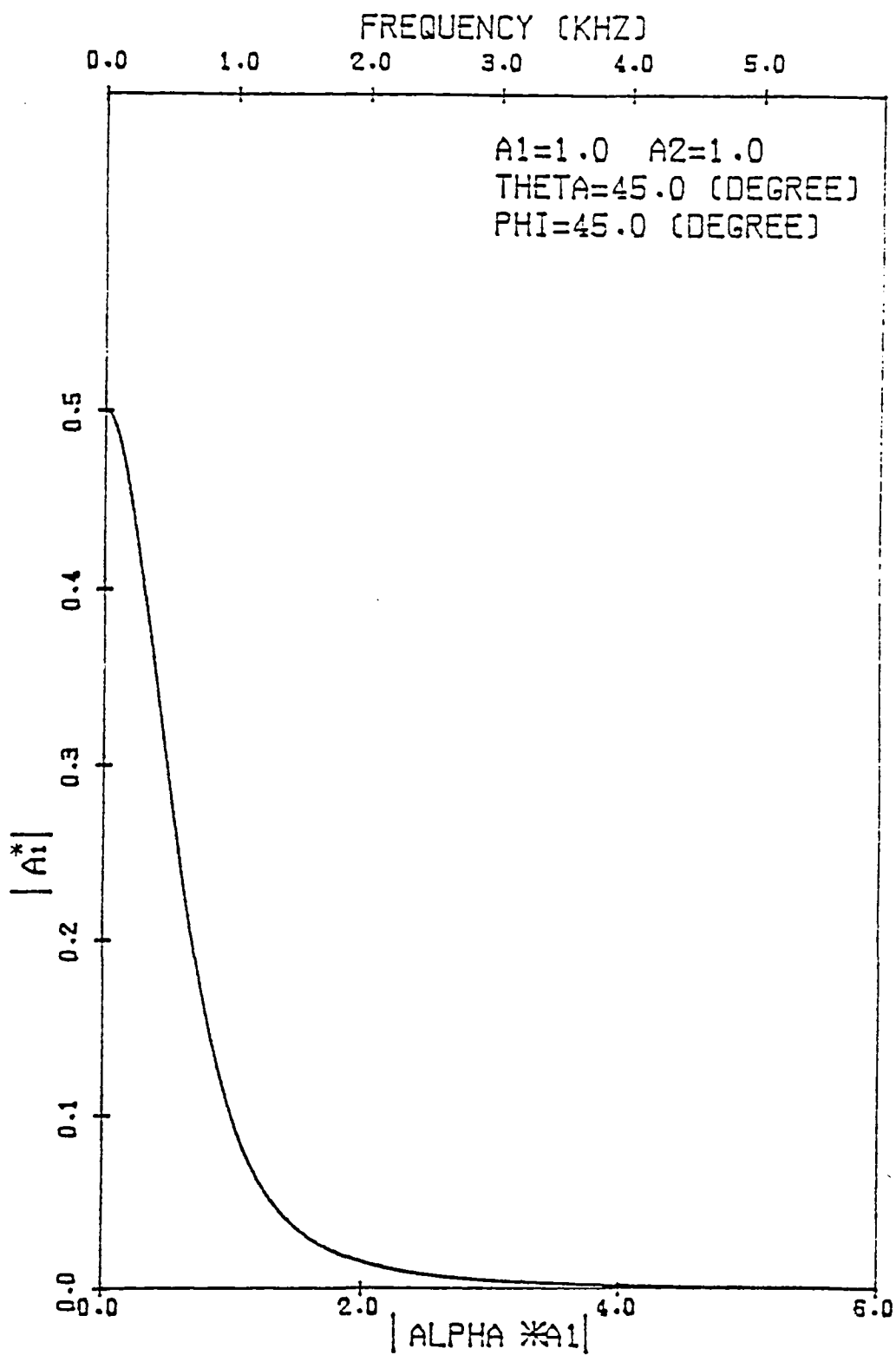


Fig. 10 Back scattered data.

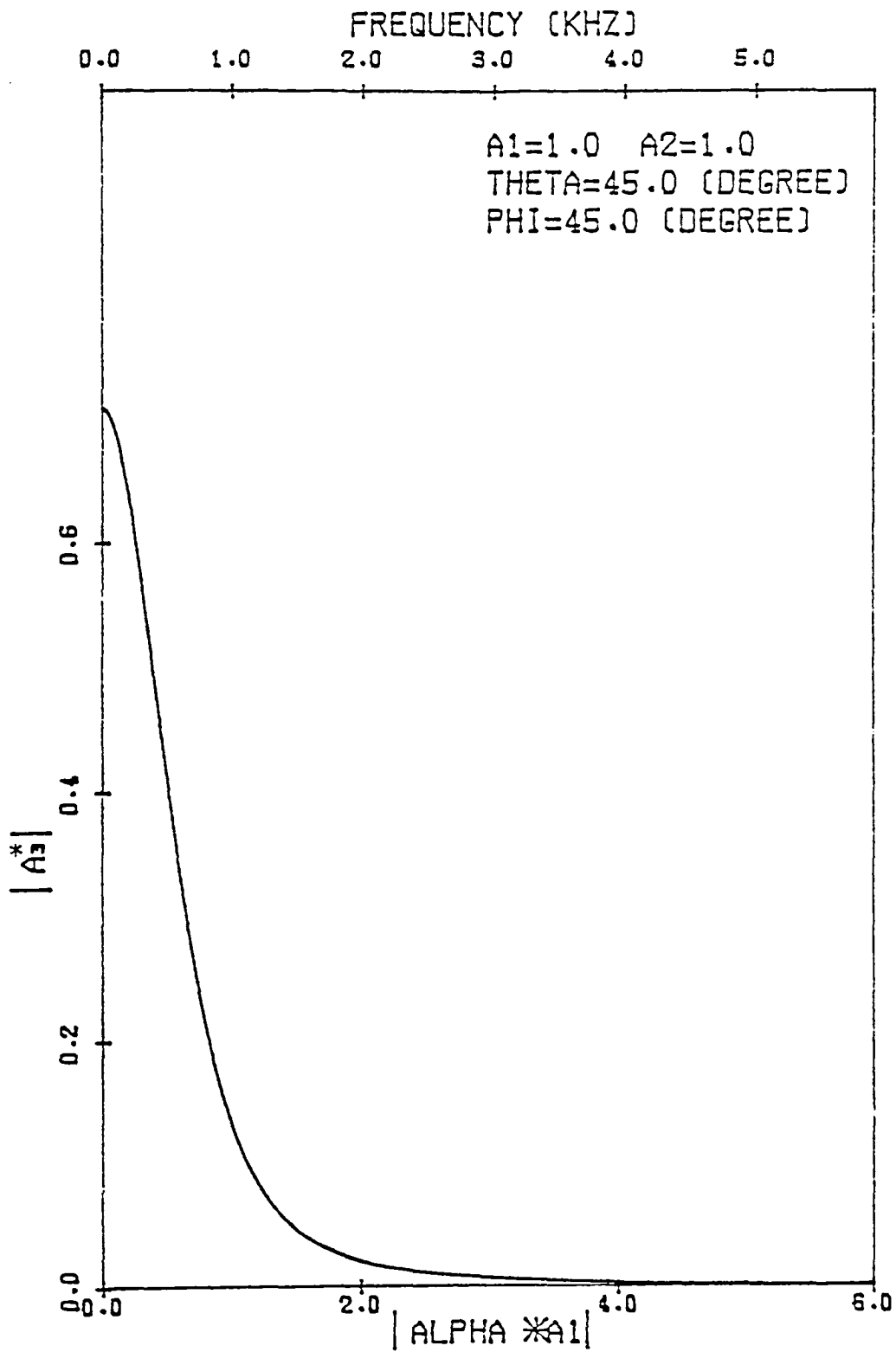


Fig. 11 Back scattered data.

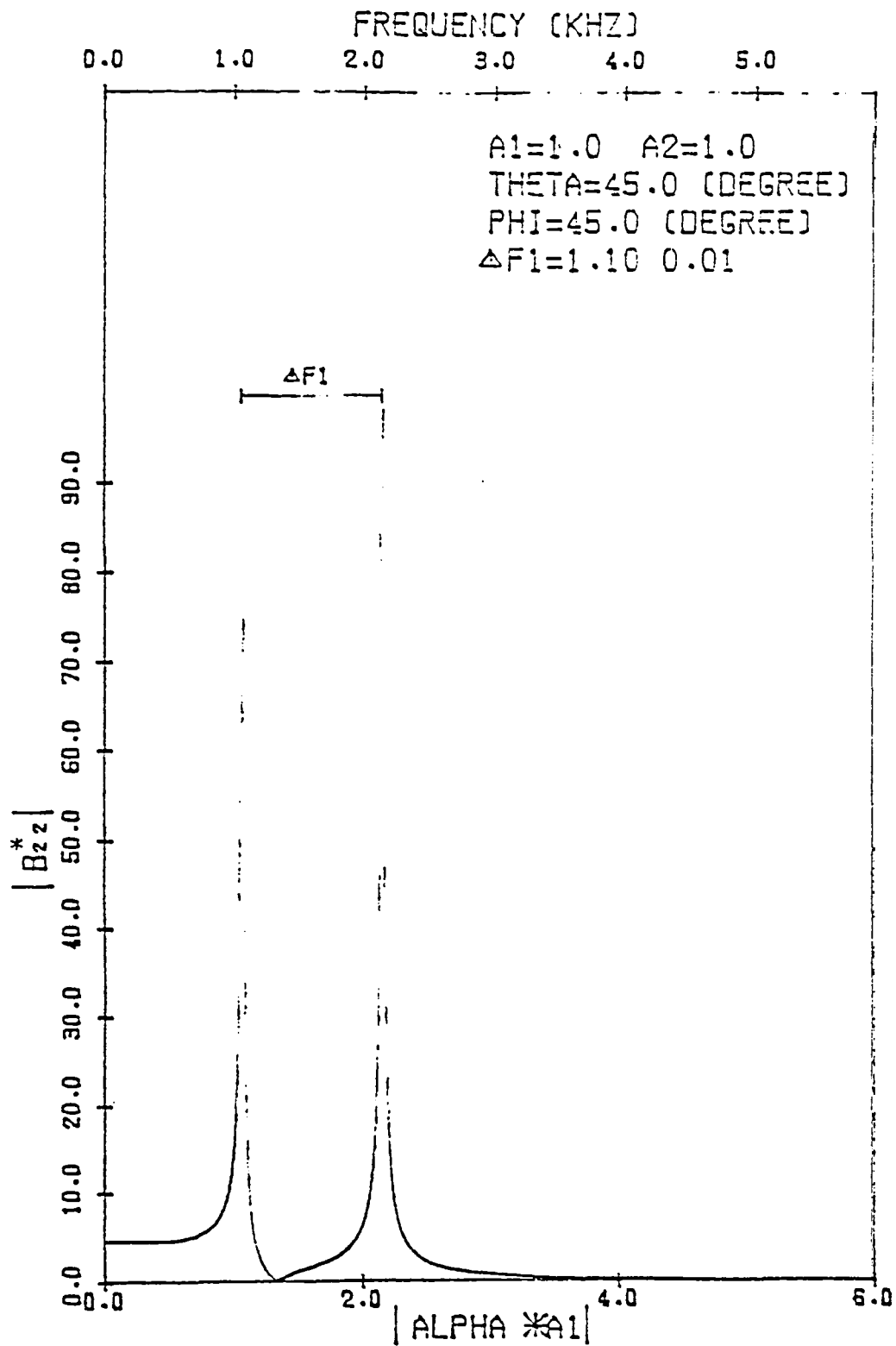


Fig. 12 Back scattered data.

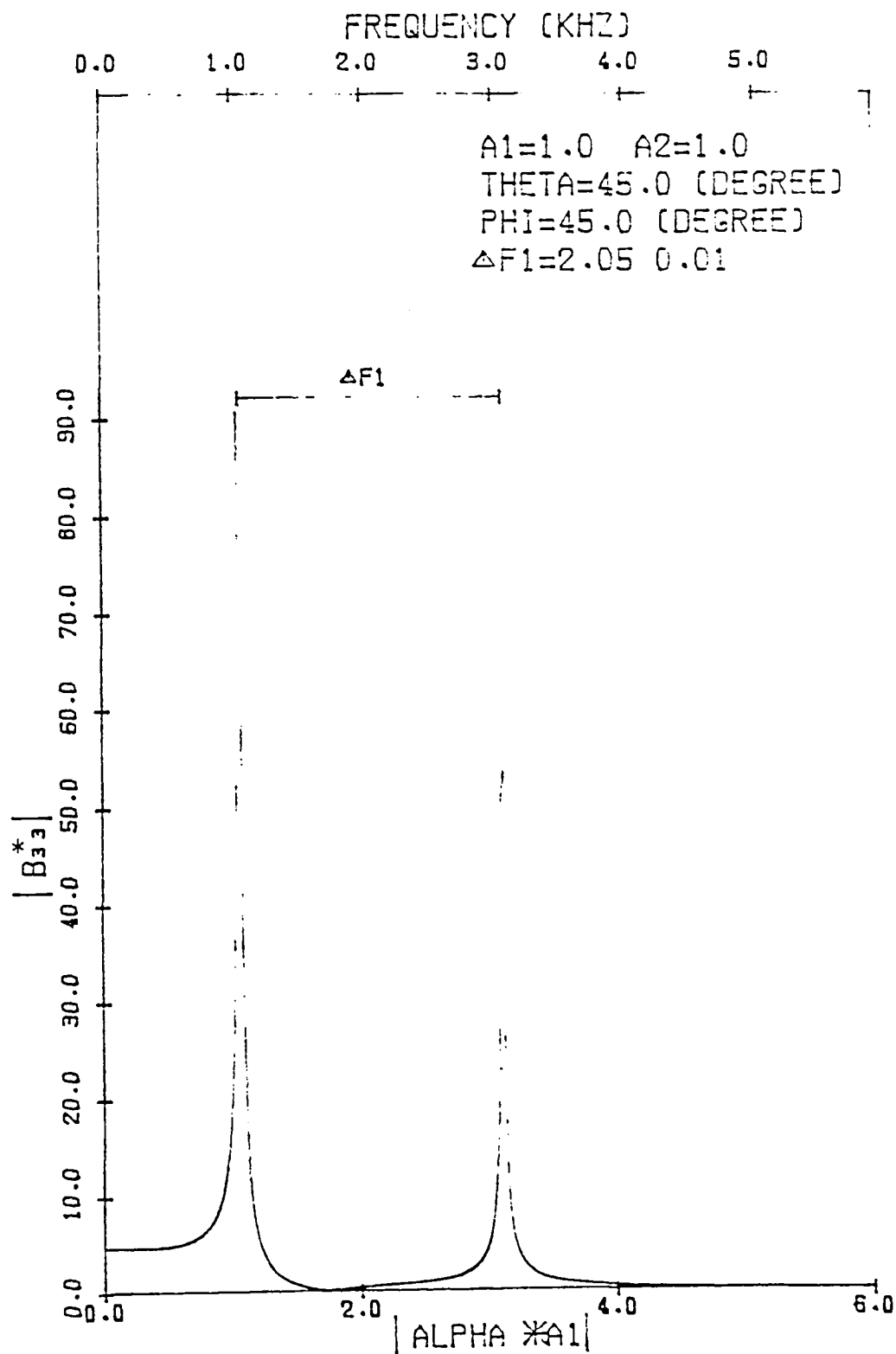


Fig. 13 Back scattered data.

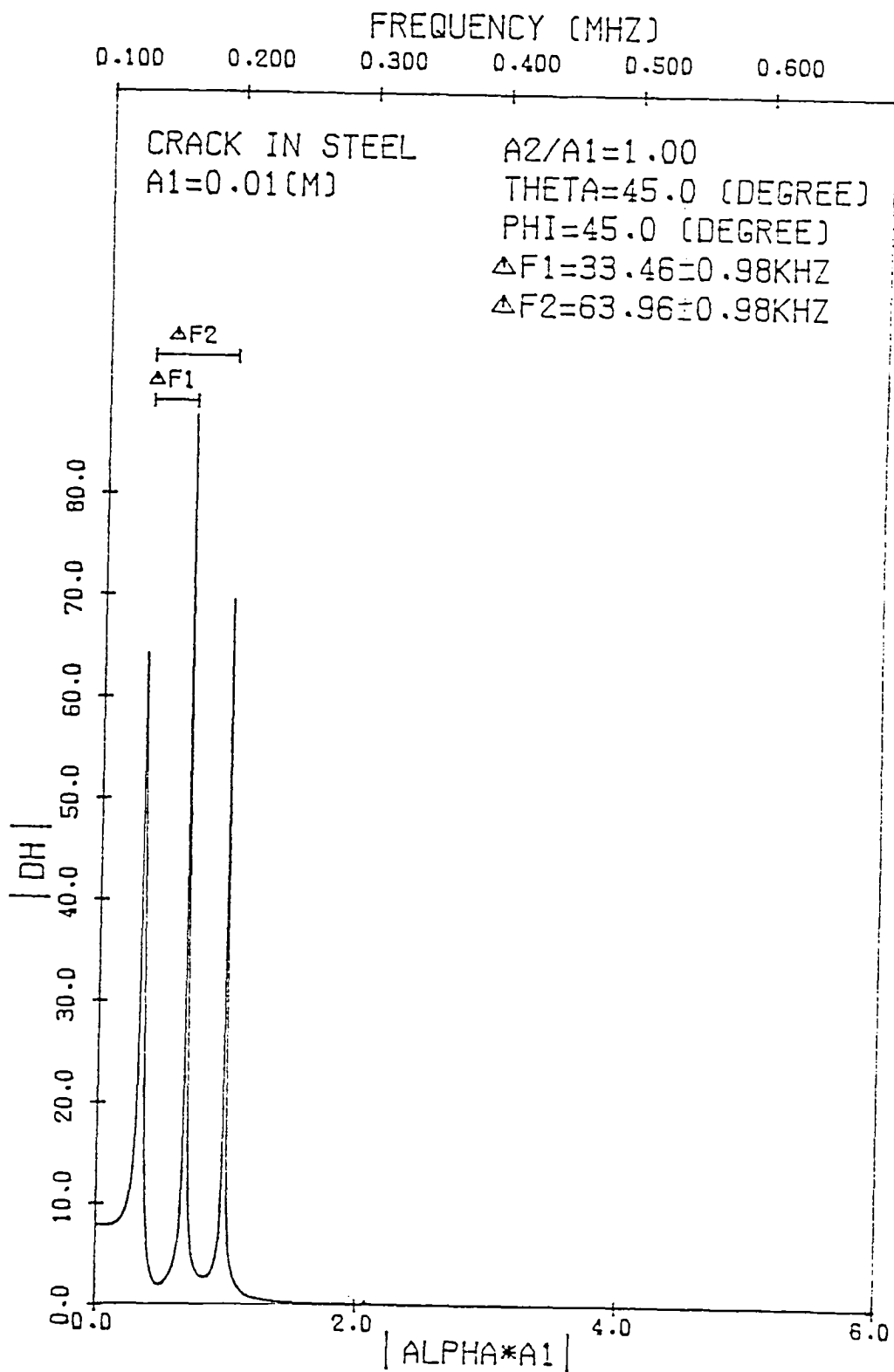


Fig. 14 Back scattered data.

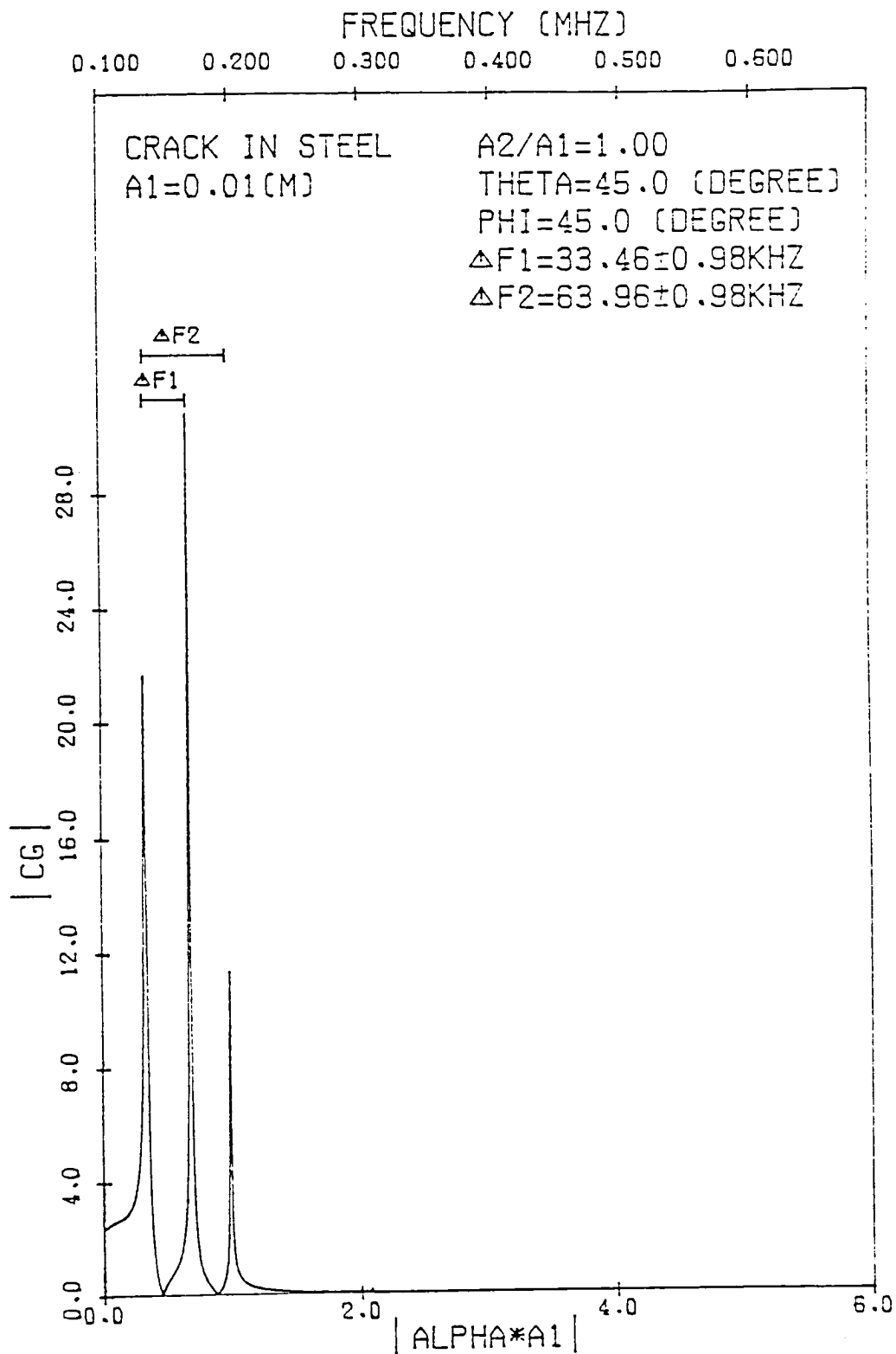


Fig. 15 Back scattered data.

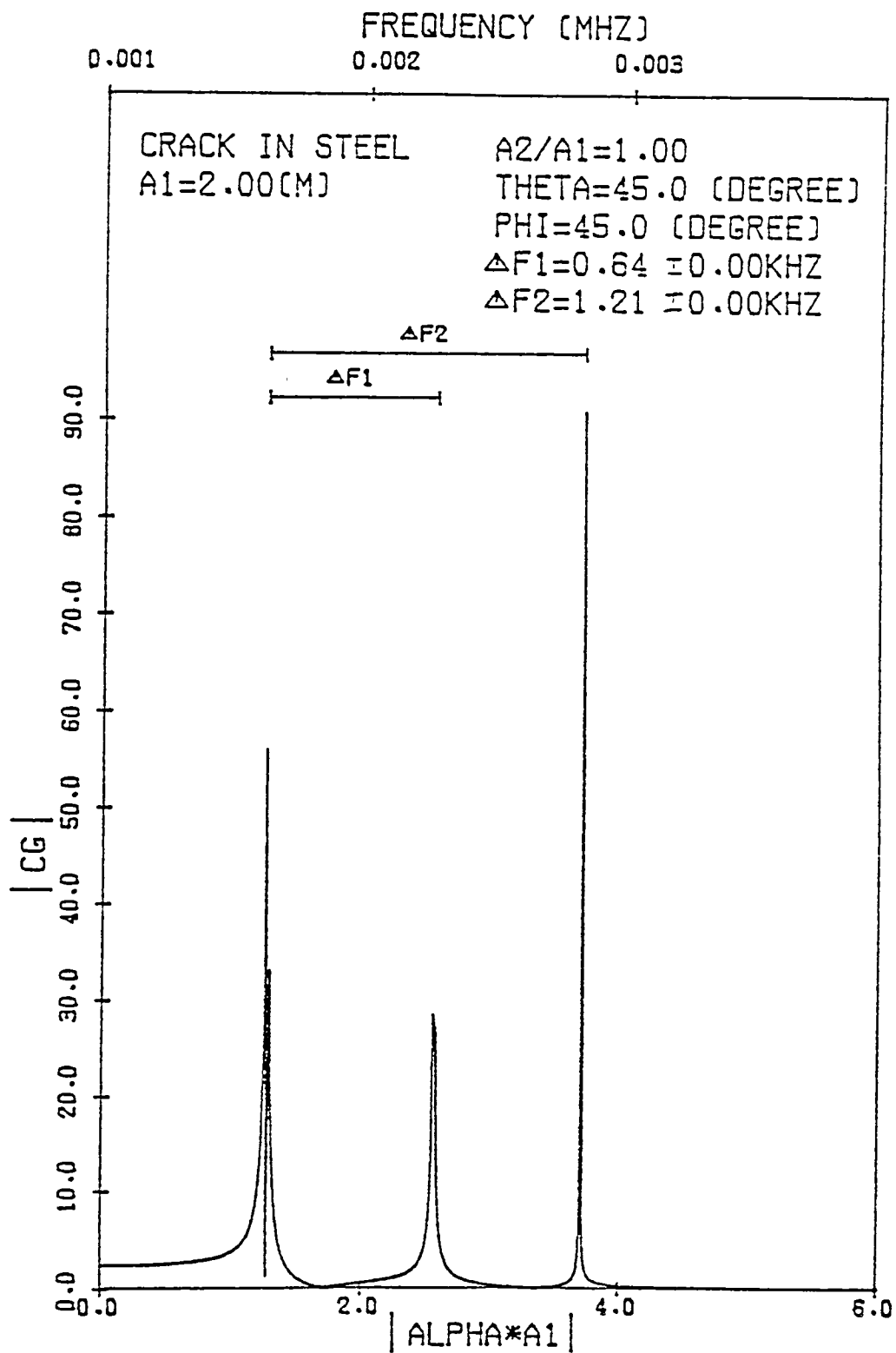


Fig. 16 Back scattered data.

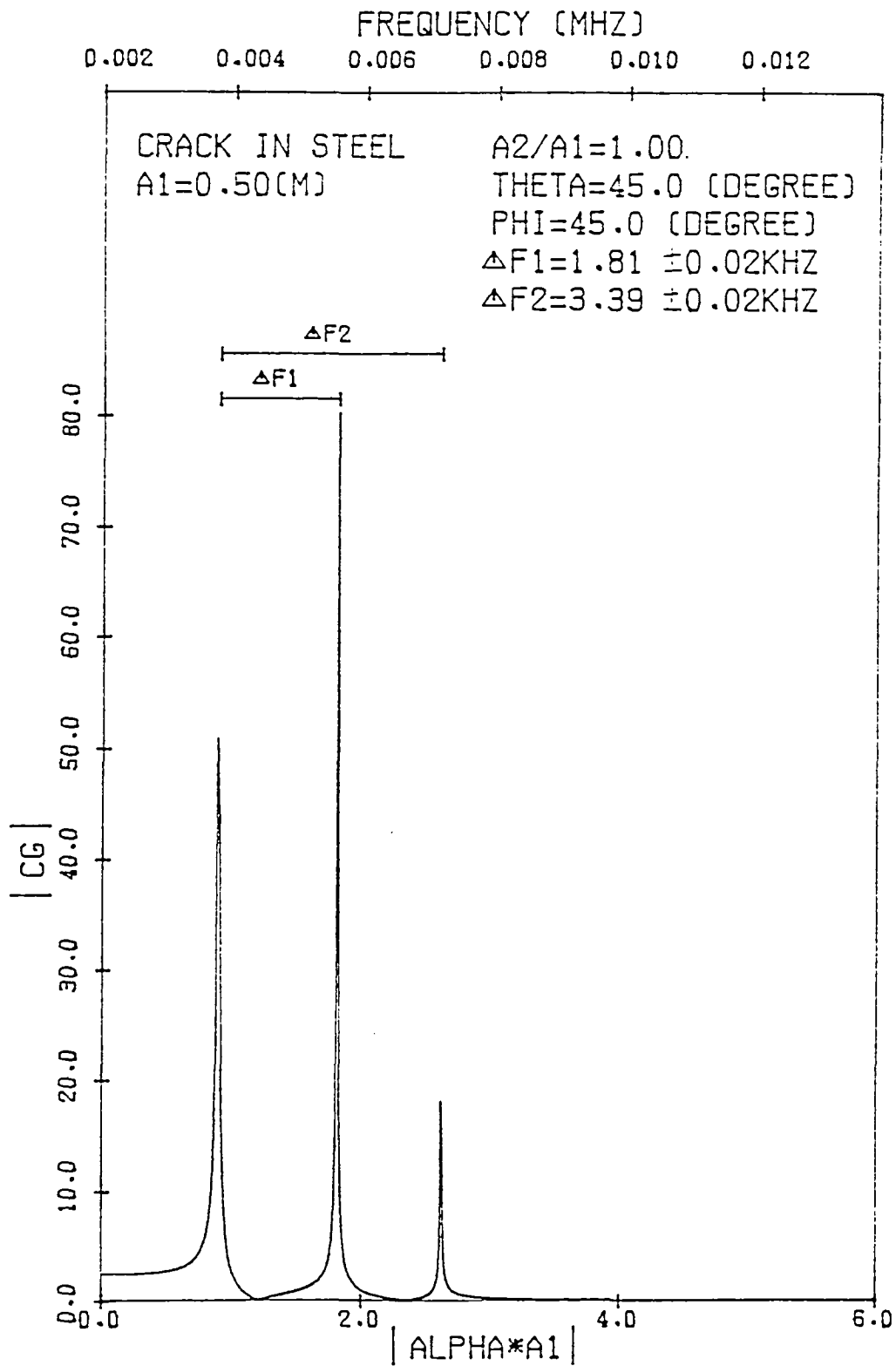


Fig. 17 Back scattered data.

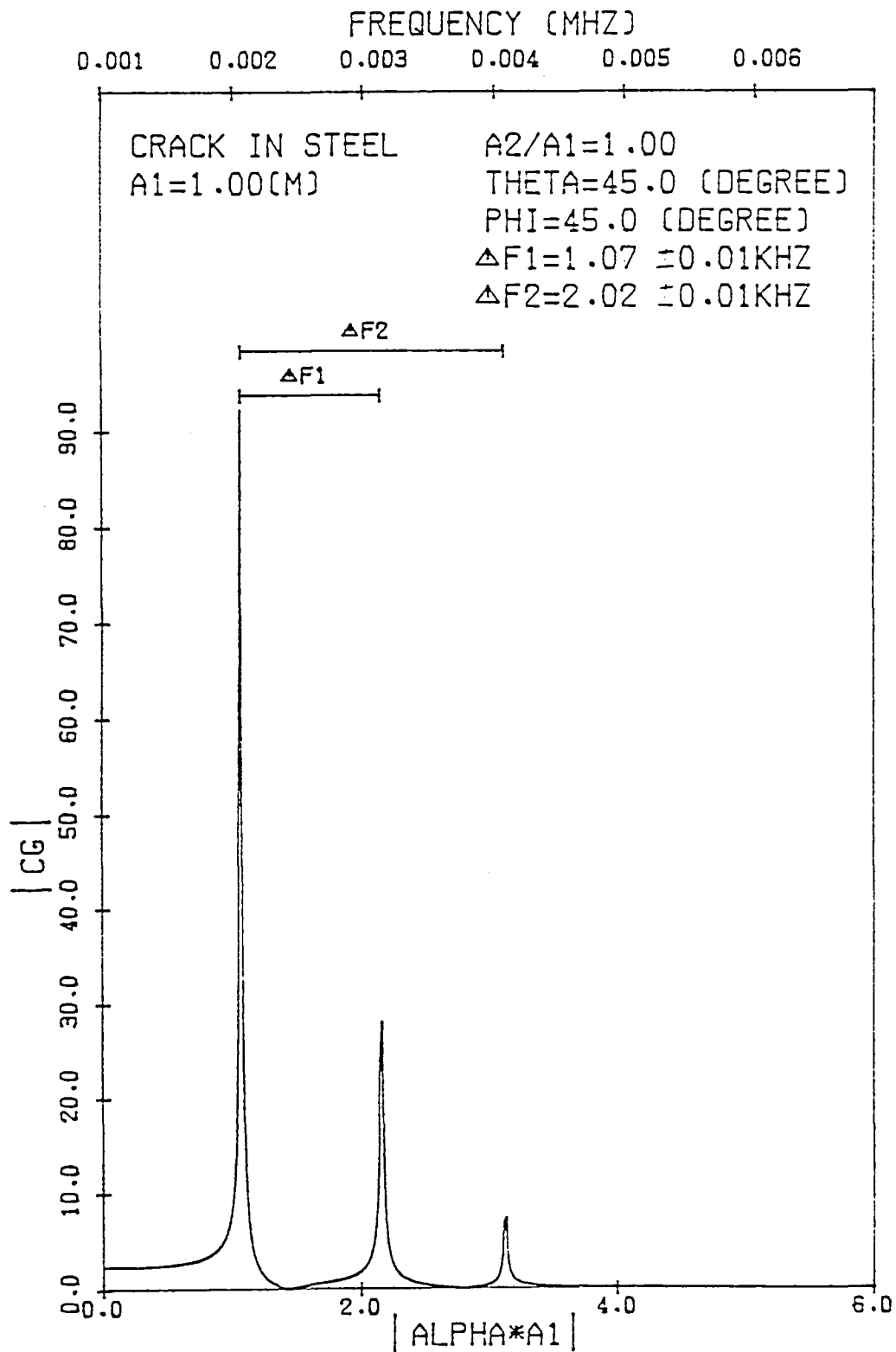


Fig. 18 Back scattered data.

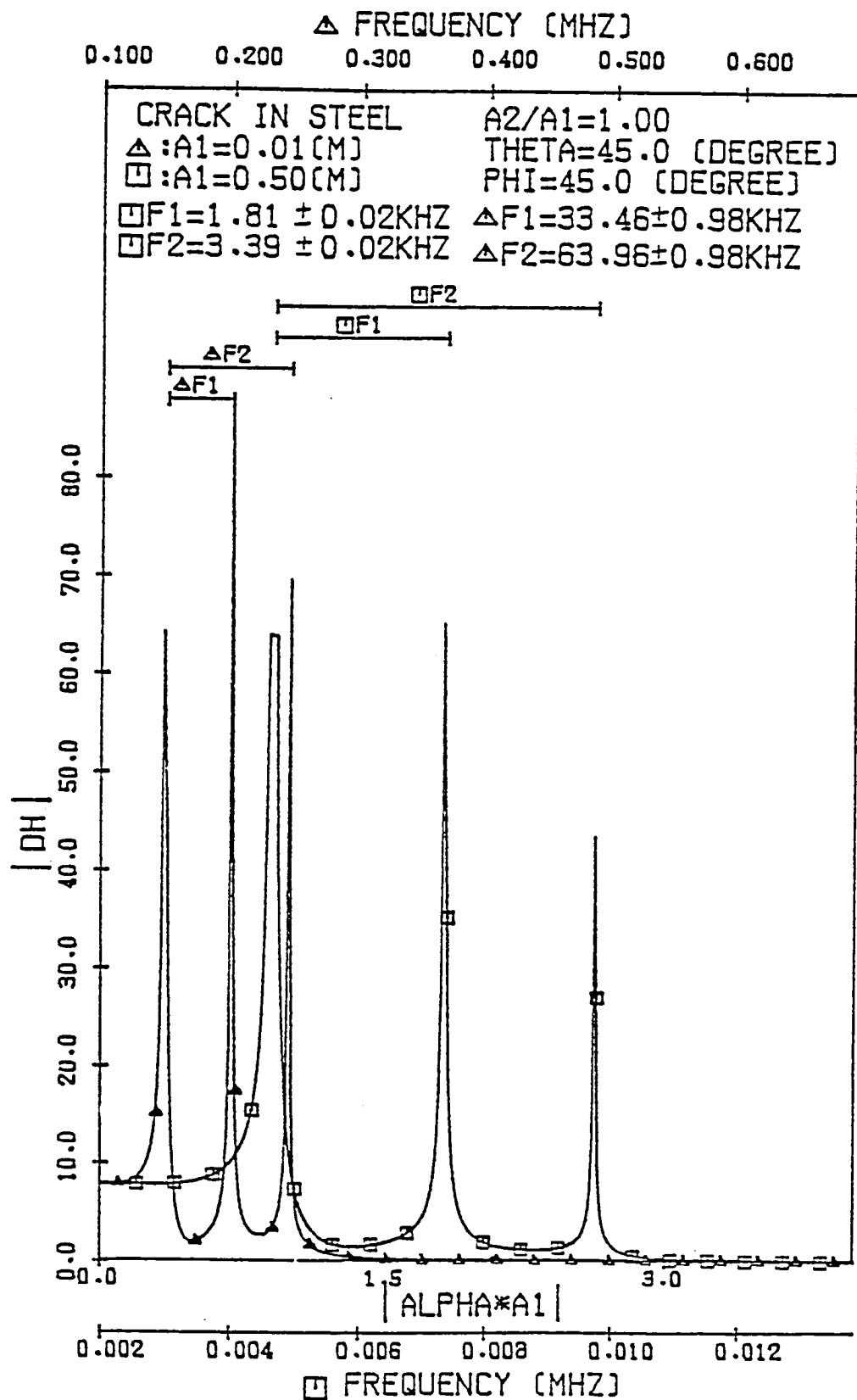


Fig. 19 Back scattered data.

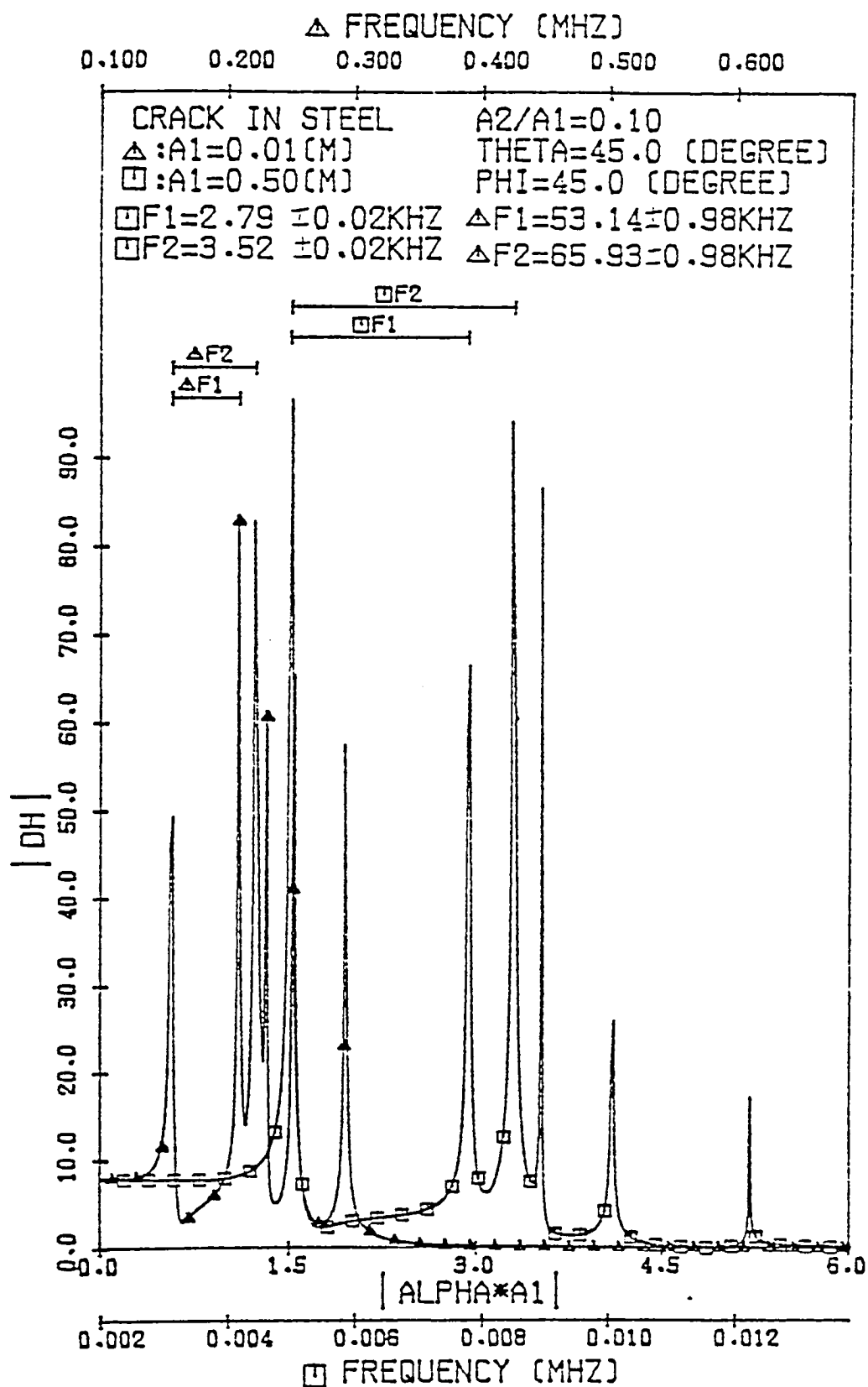


Fig. 20 Back scattered data.

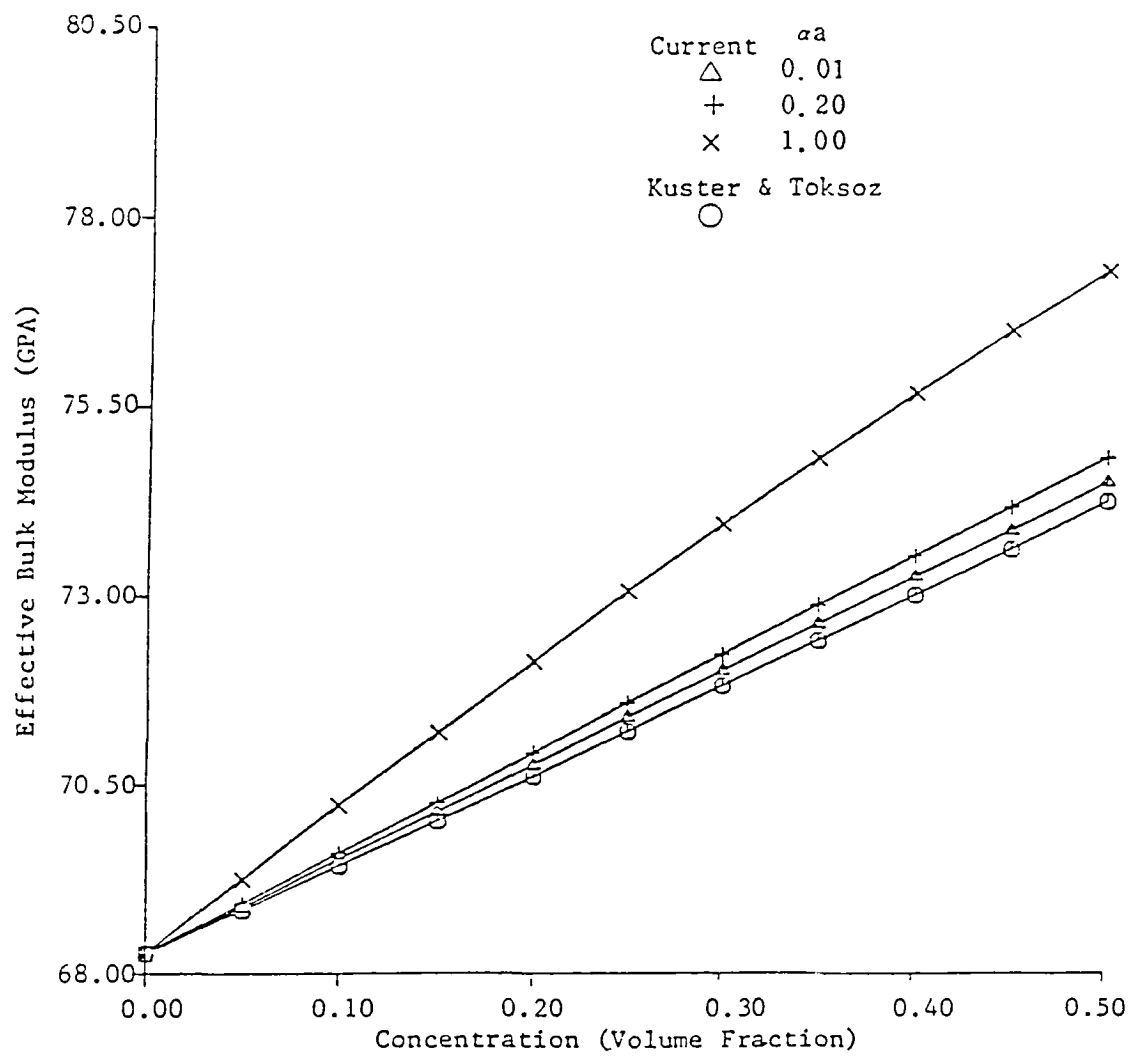


Fig. 21 Effective bulk modulus vs. concentration:
aluminum spheres in germanium.

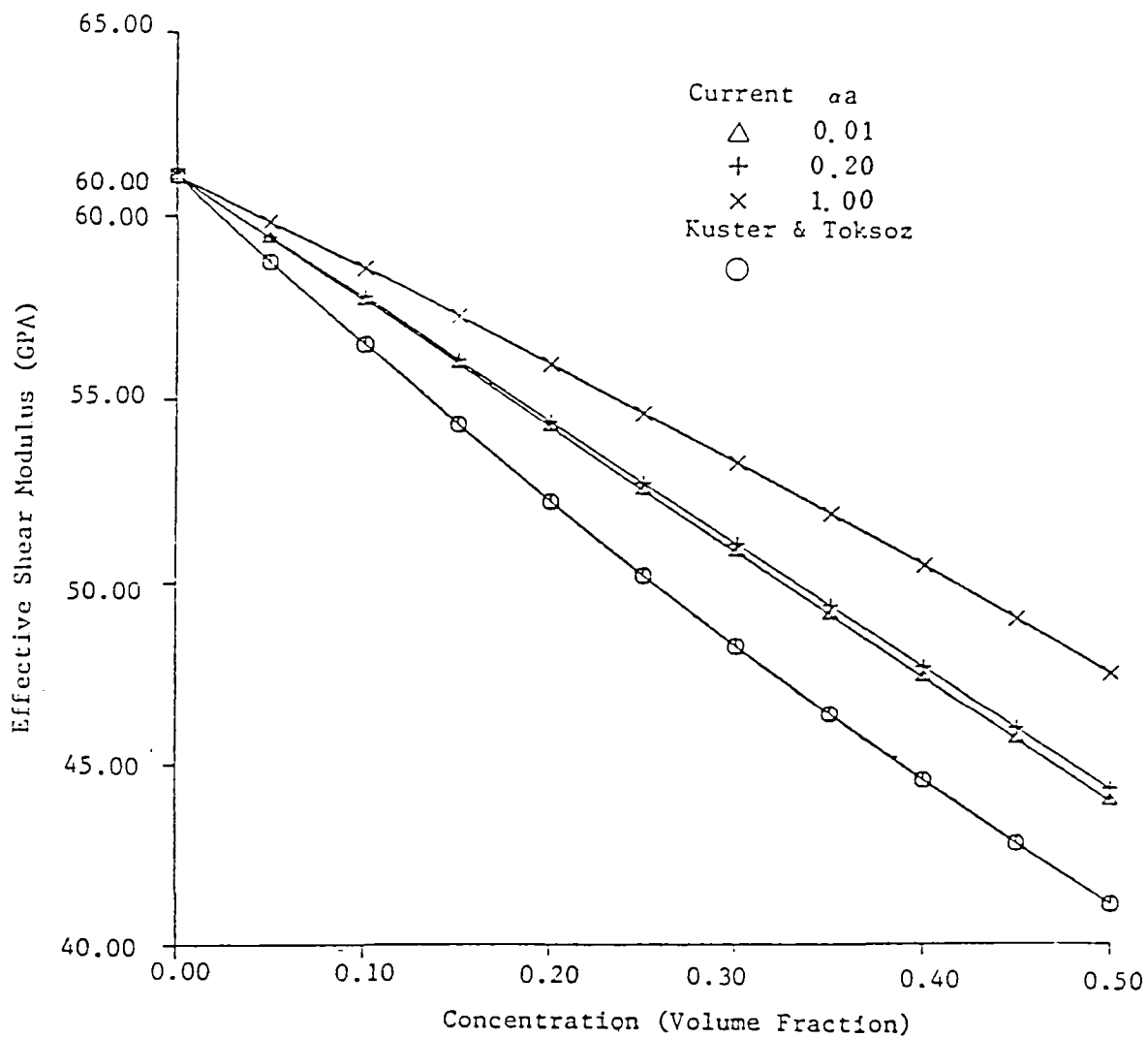


Fig. 22 Effective shear modulus vs. concentration:
aluminum spheres in germanium.

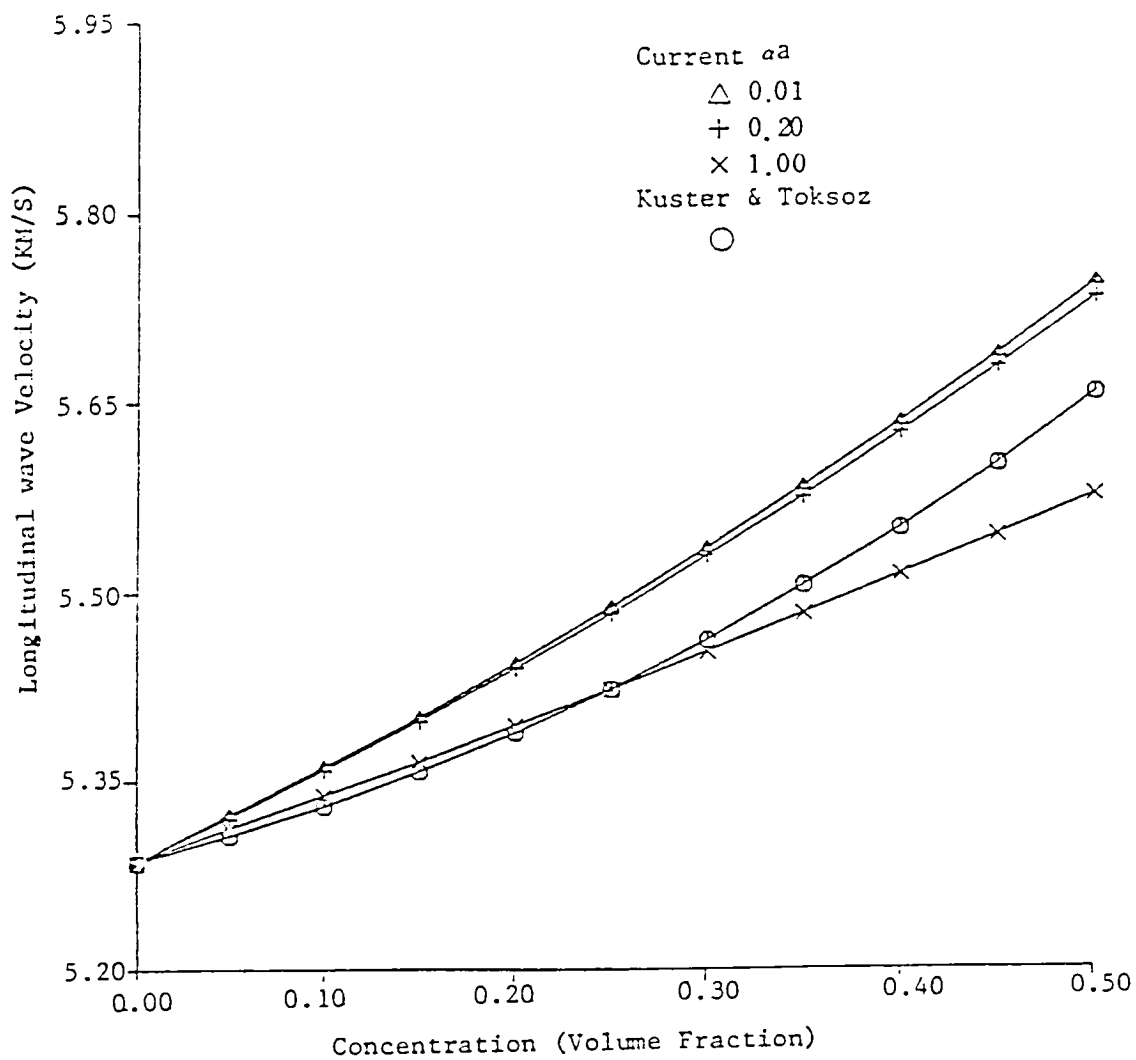


Fig. 23 Longitudinal wave velocity vs. concentration:
aluminum spheres in germanium.

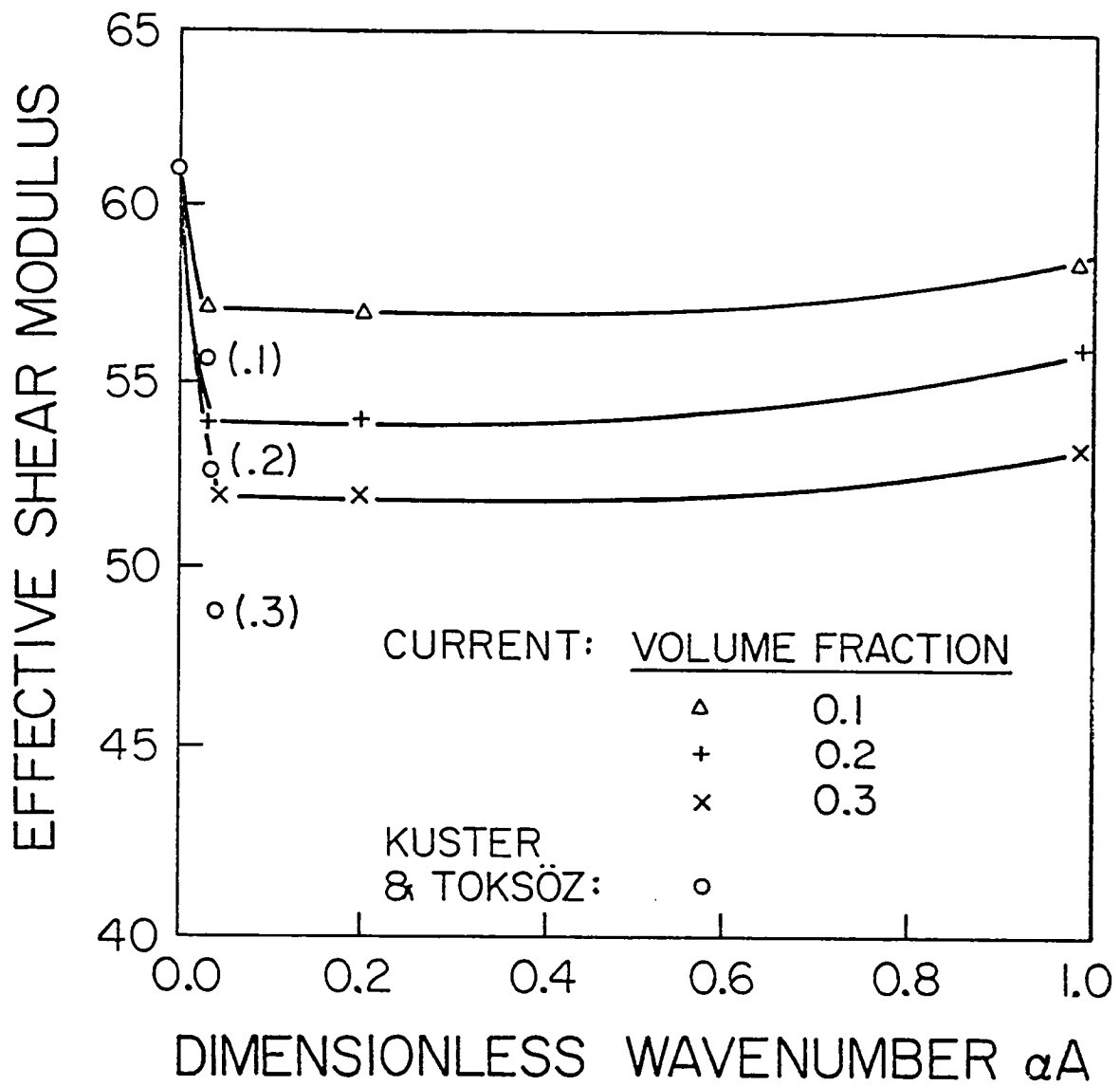


Fig. 24 Aluminum spheres in germanium

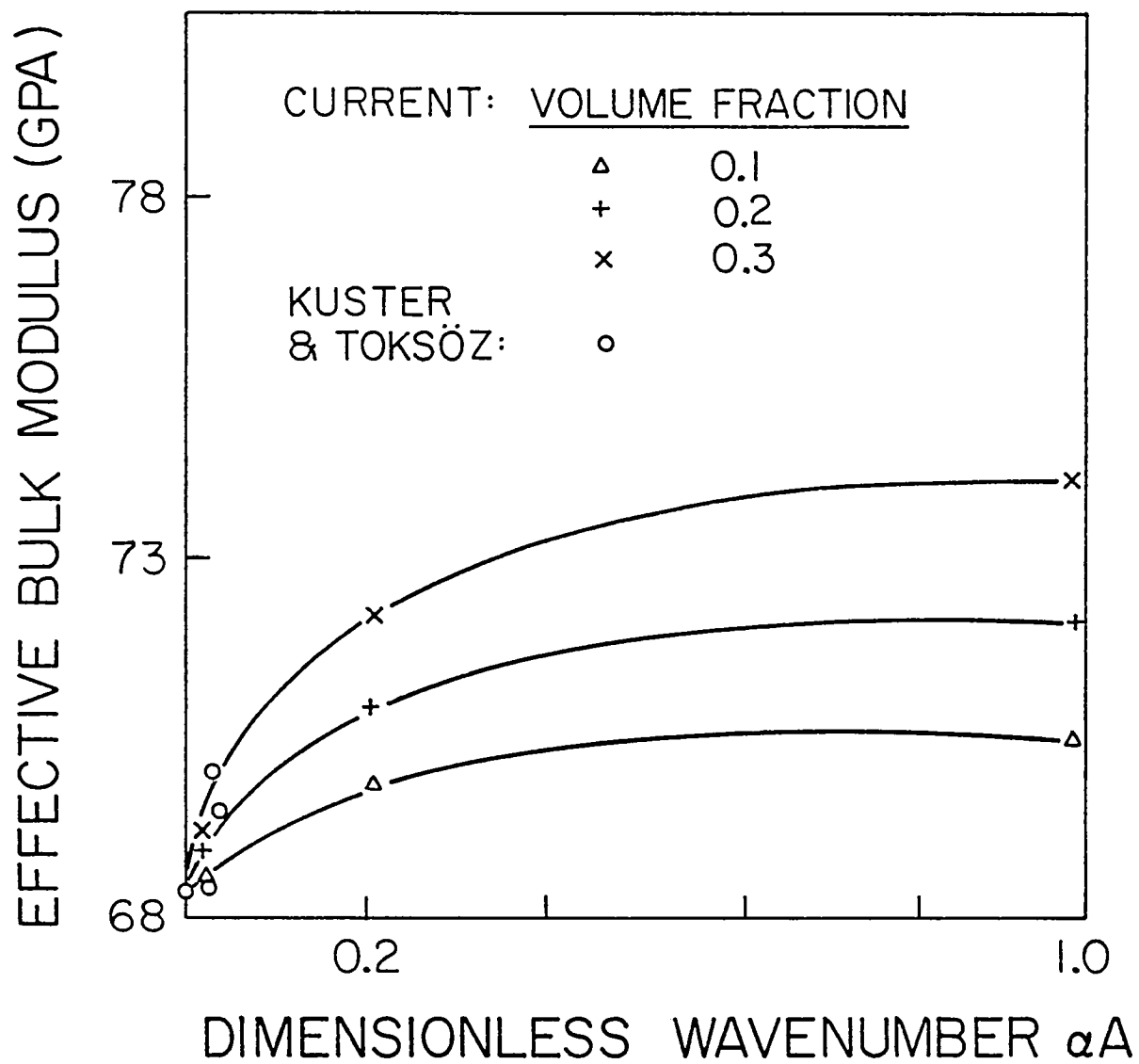


Fig. 25 Aluminum spheres in germanium.

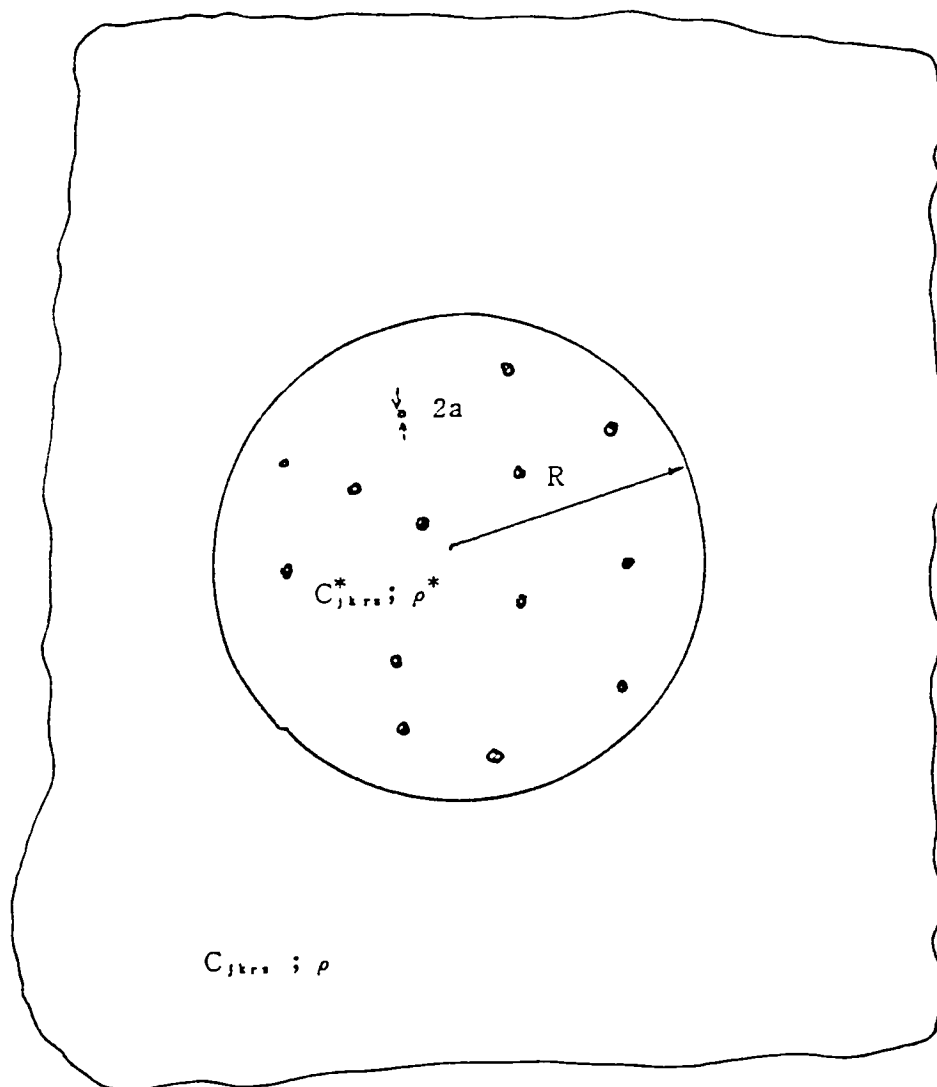


Fig. 26 A schematic diagram of localized damage in an infinite solid.

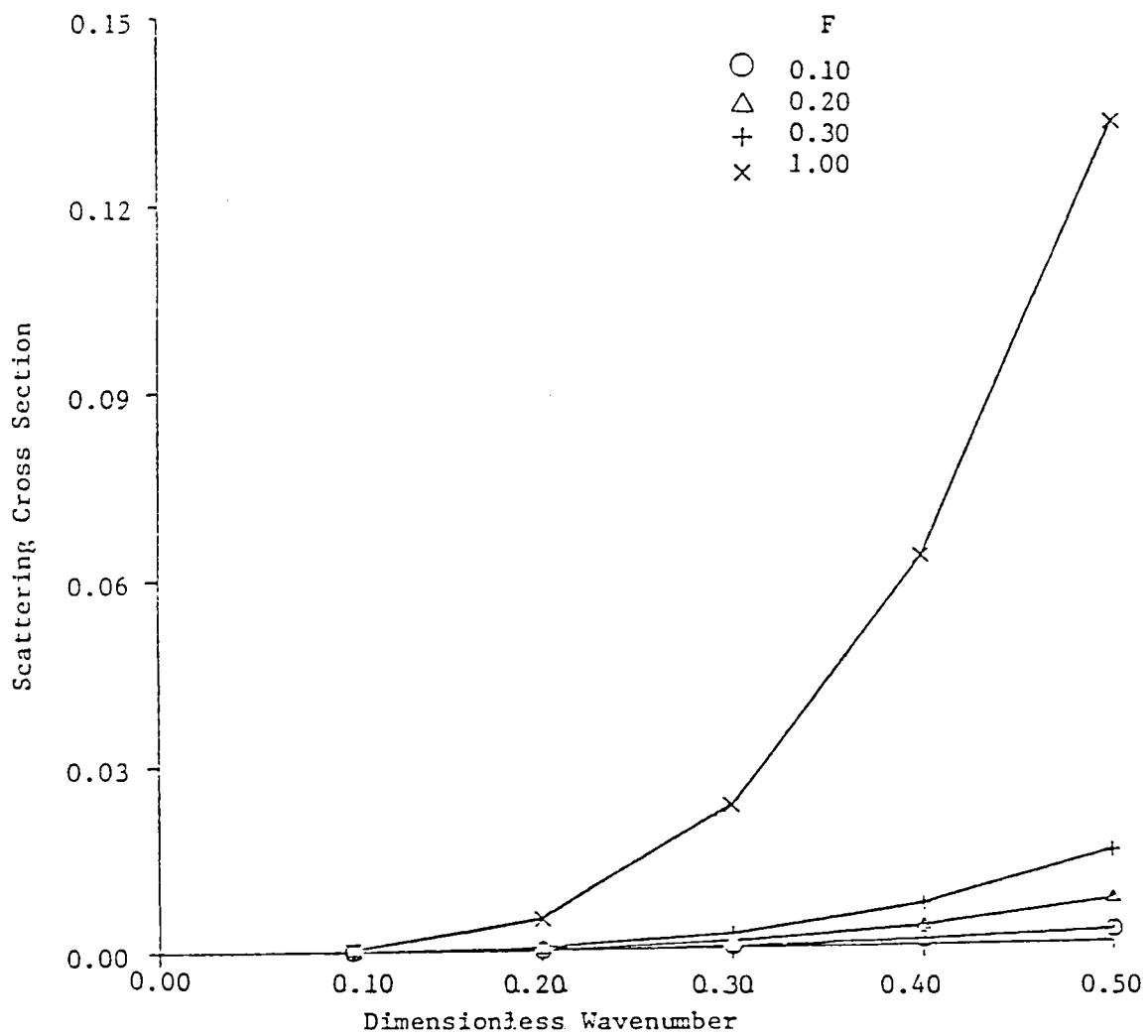


Fig. 27 Scattering cross section vs. dimensionless wavenumber: distributed voids within a spherical domain in titanium matrix, $a/R=0.01$.

APPENDIX A

Volume Integrals Associated With the Inhomogeneous Helmholtz Equation

Cylindrical Region; Rectangular Region

I. INTEGRATION OVER A FINITE CYLINDRICAL REGION, Fig. 1

The integrals, Eqs. (12,13) [1], see also [2], are of either one of the following forms:

$$\phi^O = \iiint_{\Omega} (x')^p (y')^q (z')^s dv' \quad (1)$$

$$\phi^S = \iiint_{\Omega} \rho(x', y', z') \frac{\partial^n}{\partial x'^l \partial y'^k \partial z'^{n-l-k}} \left\{ \frac{\sin \alpha r'}{r'} \right\} dv' \quad (2)$$

$$\phi^C = \iiint_{\Omega} \rho(x', y', z') \frac{\partial^n}{\partial x'^l \partial y'^k \partial z'^{n-l-k}} \left\{ \frac{\cos \alpha r'}{r'} \right\} dv' \quad (3)$$

Letting $x'=x'$, $y'=\zeta \cos \theta$, $z'=\zeta \sin \theta$ and $dv'=dx'dy'dz'=\zeta d\zeta d\theta dx'$, these integrals can be further evaluated as follows: [3]

(a)

$$\begin{aligned} \phi^O &= \iiint_{\Omega} (x')^p (y')^q (z')^s dv' \\ &= \frac{(q-1)!!(s-1)!!(2\pi)}{(q+s)!!} \left(\frac{a^{2+q+s}}{2+q+s} \right) \left(\frac{2}{1+p} \right) \quad \text{if } p, q, s \text{ all even} \end{aligned} \quad (4a)$$

$$= 0 \quad \text{if any one of } p, q, s \text{ is odd.} \quad (4b)$$

where according to the definition of factorial,

$$(q-1)!! = \frac{(q-1)!}{2^{(\frac{q}{2}-1)} (\frac{q}{2}-1)!} = 1 \cdot 3 \cdot 5 \dots (q-1), \quad q \text{ even} \quad (5a)$$

$$(q+s)!! = 2^{\frac{(q+s)}{2}} \left(\frac{q+s}{2} \right)! = 2 \cdot 4 \cdot 6 \dots s(s+2) \dots (s+q), \quad q, s \text{ even} \quad (5b)$$

and a is the radius of the cylinder, ℓ the length.

(b) $n=0$, ϕ^S :

$$\begin{aligned}
 \phi^S &= \iiint_{\Omega} (x')^p (y')^q (z')^s \frac{\sin \alpha y'}{y'} dv' \\
 &= \iiint_{\Omega} (x')^p (y')^q (z')^s \sum_{m=0}^{\infty} (-1)^m \frac{\alpha^{2m+1}}{(2m+1)!} (r')^{2m} dv' \\
 &= \sum_{m=0}^{\infty} (-1)^m \frac{\alpha^{2m+1}}{(2m+1)!} S_{m,p}
 \end{aligned} \tag{6}$$

where

$$S_{m,p} = \iiint_{\Omega} (x')^p (y')^q (z')^s (r')^{2m} dv' \tag{7}$$

Using the multinomial formula, [4]

$$(r')^{2m} = (x'^2 + y'^2 + z'^2)^m = \sum_{m_1, m_2, m_3} \frac{m!}{m_1! m_2! m_3!} (x')^{2m_1} (y')^{2m_2} (z')^{2m_3} \tag{8}$$

the integral

$$\begin{aligned}
 S_{m,p} &= \sum_{m_1, m_2, m_3} \frac{m!}{m_1! m_2! m_3!} \iiint_{\Omega} (x')^{2m_1+p} (y')^{2m_2+q} (z')^{2m_3+s} dv' \\
 &= \sum_{m_1, m_2, m_3} \frac{m!}{m_1! m_2! m_3!} \frac{(u-1)!! (v-1)!! (2\pi)}{(u+v)!!} \left(\frac{a^{2+u+v}}{2^{u+v}} \right) \left(\frac{2\ell^{1+\lambda}}{1+\lambda} \right) \\
 &\quad m=0,1,2,\dots, \text{ if } \lambda, \mu, \nu \text{ all even} \\
 &= 0 \quad \text{if any one of } \lambda, \mu, \nu \text{ is odd}
 \end{aligned} \tag{9}$$

where $(u-1)!!$, $(v-1)!!$, $(u+v)!!$ are defined as the same as (5a) and (5b),

and,

$$\begin{aligned}
 \lambda &= 2m_1 + p \\
 \mu &= 2m_2 + q \\
 \nu &= 2m_3 + s \\
 m &= m_1 + m_2 + m_3
 \end{aligned} \tag{10}$$

(c) $n=0$, ϕ^c :

$$\begin{aligned}
 \phi^c &= \iiint_{\Omega} (x')^p (y')^q (z')^s \frac{\cos \alpha r'}{r'} dv' \\
 &= \iiint_{\Omega} (x')^p (y')^q (z')^s \sum_{m=0}^{\infty} (-1)^m \frac{\alpha^{2m}}{(2m)!} \frac{(r')^{2m}}{r'} dv' \\
 &= \sum_{m=0}^{\infty} (-1)^m \frac{\alpha^{2m}}{(2m)!} C_{m,p}
 \end{aligned} \tag{11}$$

where

$$\begin{aligned}
 C_{m,p} &= \iiint_{\Omega} (x')^p (y')^q (z')^s \frac{(r')^{2m}}{r'} dv' \\
 &= \sum_{m_1, m_2, m_3} \frac{m!}{m_1! m_2! m_3!} \iiint_{\Omega} (x')^{\lambda} (y')^{\mu} (z')^{\nu} \frac{dv'}{r'}
 \end{aligned} \tag{12}$$

$$\lambda = 2m_1 + p$$

$$\mu = 2m_2 + q$$

$$\nu = 2m_3 + s$$

$$m_1 + m_2 + m_3 = m \tag{13}$$

The integral on the right hand side of Eq.(12) can be evaluated as follows:

$$\begin{aligned}
 &\iiint_{\Omega} (x')^{\lambda} (y')^{\mu} (z')^{\nu} \frac{dv'}{r'} = \\
 &= \frac{(\mu-1)!!(\nu-1)!!}{(\mu+\nu)!!} (2\pi) \cdot I \quad \text{if } \mu, \nu \text{ all even}
 \end{aligned} \tag{14a}$$

$$= 0 \quad \text{if any one of } \mu, \nu \text{ is odd.} \tag{14b}$$

In the above expression

$$\begin{aligned}
 I &= \frac{1}{k+1} \sum_{t=1}^{\frac{1}{2}(k-1)} (-1)^{t+1} \frac{(k+1)(k-1)\dots(k-2t+3)}{k(k-2)\dots(k-2t+2)} (a)^{k-2t+1} \int_{-\ell}^{\ell} (x')^{\lambda+2t-2} (a^2+x'^2)^{1/2} dx' \\
 &+ (-1)^{\frac{1}{2}(k-1)} \frac{(k-1)!!}{k!!} \int_{-\ell}^{\ell} (x')^{\lambda+k-1} (a^2+x'^2)^{1/2} dx'
 \end{aligned} \tag{15}$$

where

$$k = 1 + \mu + \nu \quad (16)$$

is odd, and

$$(k-1)!! = 2^{\frac{(k-1)}{2}} \cdot \left(\frac{k-1}{2}\right)! = 2 \cdot 4 \dots (k-1) \quad (17a)$$

$$(k)!! = \frac{k!}{2^{\frac{(k-1)}{2}} \left(\frac{k-1}{2}\right)!} = 1 \cdot 3 \dots (k) \quad (17b)$$

The integrals

$$\begin{aligned} & \int_{-l}^l (x')^{\lambda+2t-2} (a^2+x'^2)^{1/2} dx' \\ &= \frac{2}{k_1+1} \sum_{t_1=1}^{\frac{k_1}{2}} (-1)^{t_1+1} \frac{(k+1)(k_1-1)\dots(k_1-2t_1+3)}{(k_1+2)k_1\dots(k_1-2t_1+4)} (a)^{2t_1-2} \cdot (l)^{k_1-2t_1+1} \cdot (a^2+l^2)^{3/2} \\ &+ (-1)^{\frac{k_1}{2}} \frac{(k_1-1)!!}{(k_1+2)(k_1)!!} 2a^{k_1} (l\sqrt{a^2+l^2} + a^2 \operatorname{sh}^{-1} \frac{l}{a}) \quad \text{if } \lambda, \mu, \nu \text{ are even} \\ &= 0 \quad \text{if any one of } \lambda, \mu, \nu \text{ is odd.} \end{aligned} \quad (18)$$

where

$$k_1 = \lambda + 2t - 2 \quad (19)$$

is even, and

$$(k_1-1)!! = \frac{(k_1-1)!}{2^{\frac{(k_1-1)}{2}} \left(\frac{k_1-1}{2}\right)!} = 1 \cdot 3 \cdot 5 \dots (k_1-1) \quad (20a)$$

$$(k_1)!! = 2^{\frac{k_1}{2}} \left(\frac{k_1}{2}\right)! = 2 \cdot 4 \dots (k_1) \quad k_1 \text{ is even} \quad (20b)$$

The integral

$$\begin{aligned} & \int_{-l}^l (x')^{\lambda+k-1} (a^2+x'^2)^{1/2} dx' \\ &= \int_{-l}^l (x')^{k_2} (a^2+x'^2)^{1/2} dx' \end{aligned}$$

$$= \frac{2}{k_2+1} \sum_{\tau_2=1}^{\frac{k_2}{2}} (-1)^{\tau_2+1} \frac{(k_2+1)(k_2-1)\dots(k_2-2\tau_2+3)}{(k_2+2)k_2\dots(k_2-2\tau_2+4)} (a)^{2\tau_2-2} \cdot (\ell)^{k_2-2\tau_2+1} \cdot (a^2+\ell^2)^{3/2}$$

$$= (-1)^{\frac{k_2}{2}} \frac{(k_2-1)!!}{(k_2+2) \cdot (k_2)!!} \cdot 2a^{k_2} (\ell \sqrt{a^2+\ell^2} + a^2 \operatorname{sh}^{-1} \frac{\ell}{a}) \quad \text{if } \lambda, \mu, \nu \text{ are even} \quad (21a)$$

$$= 0 \quad \text{if any one of } \lambda, \mu, \nu \text{ is odd.} \quad (21b)$$

where

$$k_2 = \lambda + k - 1 = \lambda + \mu + \nu \quad (22)$$

The integral

$$\int_{-\ell}^{\ell} (x')^{\lambda+k} dx' = \frac{2\ell^{\lambda+k+1}}{\lambda+k+1} \quad \text{if } \lambda, \mu, \nu \text{ are even} \quad (23a)$$

$$= 0 \quad \text{if any one of } \lambda, \mu, \nu \text{ is odd.} \quad (23b)$$

II. INTEGRATION OVER A RECTANGULAR PARALLELEPIPED, Fig. 2

In the case of a rectangular parallelepiped, the integrals in (14), (15), (16) can be evaluated as follows:

(a)

$$\begin{aligned}\phi^0 &= \iiint_{\Omega} (x')^p (y')^q (z')^s dv' \\ &= \frac{8}{(p+1)(q+1)(s+1)} (a)^{q+1} (b)^{s+1} (\ell)^{p+1} \quad \text{if } p, q, s \text{ all are even} \quad (24a)\end{aligned}$$

$$= 0 \quad \text{if any one of } p, q, s \text{ is odd.} \quad (24b)$$

where ℓ, a, b is length of the rectangular parallelepiped toward x', y', z' -direction, respectively.

(b) $n=0, \phi^s$

$$\begin{aligned}\phi^s &= \iiint_{\Omega} (x')^p (y')^q (z')^s \frac{\sin ar'}{r'} dv' \\ &= \iiint_{\Omega} (x')^p (y')^q (z')^s \sum_{m=0}^{\infty} (-1)^m \frac{a^{2m+1}}{(2m+1)!} (r')^{2m} \\ &= \sum_{m=0}^{\infty} (-1)^m \frac{a^{2m+1}}{(2m+1)!} S_{m,p} \quad (25)\end{aligned}$$

and

$$\begin{aligned}S_{m,p} &= \iiint_{\Omega} (x')^p (y')^q (z')^s (r')^{2m} dv' \\ &= \sum_{m_1, m_2, m_3} \frac{m!}{m_1! m_2! m_3!} \iiint_{\Omega} (x')^{2m_1+p} (y')^{2m_2+q} (z')^{2m_3+s} dv' \\ &= \sum_{m_1, m_2, m_3} \frac{8m!}{m_1! m_2! m_3!} \frac{(a)^{\mu+1} (b)^{\nu+1} (\ell)^{\lambda+1}}{(\lambda+1)(\mu+1)(\nu+1)} \\ &\quad m=0, 1, 2, \dots \quad \text{if } \lambda, \mu, \nu \text{ all are even} \quad (26a) \\ &= 0 \quad \text{if any one of } \lambda, \mu, \nu \text{ is odd.} \quad (26b)\end{aligned}$$

where

$$\begin{aligned}
m &= m_1 + m_2 + m_3 \\
\lambda &= 2m_1 + p \\
\mu &= 2m_2 + q \\
\nu &= 2m_3 + s
\end{aligned} \tag{27}$$

(c) $n=0$, ϕ^C .

$$\begin{aligned}
\phi^C &= \iiint_{\Omega} (x')^p (y')^q (z')^s \frac{\cos \alpha r'}{r'} dv' \\
&= \iiint_{\Omega} (x')^p (y')^q (z')^s \sum_{m=0}^{\infty} (-1)^m \frac{\alpha^{2m}}{(2m)!} \frac{(r')^{2m}}{r'} \\
&= \sum_{m=0}^{\infty} (-1)^m \frac{\alpha^{2m}}{(2m)!} C_{m,p}
\end{aligned} \tag{28}$$

and

$$\begin{aligned}
C_{m,p} &= \sum_{m_1, m_2, m_3} \frac{m!}{m_1! m_2! m_3!} \iiint_{\Omega} (x')^{\lambda} (y')^{\mu} (z')^{\nu} \frac{dv'}{r'} \\
&= \sum_{m_1, m_2, m_3} \frac{m!}{m_1! m_2! m_3!} \int_{-\ell}^{\ell} (x')^{\lambda} \left[\int_{-a}^a (y')^{\mu} \left(\int_{-b}^b \frac{(z')^{\nu}}{\sqrt{\alpha^2 + z'^2}} dz' \right) dy' \right] dx'
\end{aligned} \tag{29}$$

where

$$\begin{aligned}
\lambda &= 2m_1 + p \\
\mu &= 2m_2 + q \\
\nu &= 2m_3 + s \\
\alpha^2 &= x'^2 + y'^2 \\
m_1 + m_2 + m_3 &= m
\end{aligned} \tag{30}$$

The integrals in (34) can be evaluated as follows:

$$\begin{aligned}
&\text{(i) } \int_{-b}^b \frac{(z')^{\nu}}{\sqrt{\alpha^2 + z'^2}} dz' \\
&= \Pi_{\nu} (b)^{\nu-2t+1} (x'^2 + y'^2)^{\nu/2} \sqrt{b^2 + x'^2 + y'^2} + (-1)^{\nu/2} \frac{2(\nu-1)!!}{(\nu)!!} (x'^2 + y'^2)^{\nu/2} \text{sh}^{-1} \frac{b}{\sqrt{x'^2 + y'^2}}
\end{aligned}$$

$$\text{if } v \text{ is even} \quad (31a)$$

$$=0 \quad \text{if } v \text{ is odd.} \quad (31b)$$

where Π_v is defined as an operator as follows:

$$\Pi_v = \frac{2}{v+1} \sum_{t=1}^{v/2} (-1)^{t+1} \frac{(v+1)(v-1)\dots(v-2t+3)}{v(v-2)(v-4)\dots(v-2t+2)} \quad (32a)$$

$$(v-1)!! = \frac{(v-1)!}{2^{(\frac{v}{2}-1)} (\frac{v}{2}-1)!} = 1 \cdot 3 \dots (v-1) \quad (32b)$$

$$(v)!! = 2^{\frac{v}{2}} (\frac{v}{2})! = 2 \cdot 4 \dots v, \quad v \text{ is even} \quad (32c)$$

(ii)

$$\begin{aligned} & \int_{-a}^a \int_{-b}^b (y')^\mu (z')^\nu \frac{dy' dz'}{\sqrt{x'^2 + y'^2 + z'^2}} \\ &= \Pi_v \sum_{n_1, n_2} (-1)^{v/2} \frac{n!}{n_1! n_2!} (b)^{v-2t+1} (x')^{2n_1} I_1(x') \\ &+ \sum_{i_1, i_2} (-1)^{v/2} \frac{i!}{i_1! i_2!} \frac{2(v-1)!!}{(v)!!} (b)(x')^{2i_1} I_2(x') \\ &- \sum_{i_1, i_2} (-1)^{v/2} \frac{i!}{i_1! i_2!} \frac{(v-1)!!}{3(v)!!} (b)^3 (x')^{2i_1} I_3(x') \\ &+ \sum_{i_1, i_2} \sum_{j=2}^{\infty} (-1)^{v/2+j} \frac{i!}{i_1! i_2!} \frac{2(v-1)!!}{(v)!!} \frac{(2j)!}{2^{2j} (j!)^2 (2j+1)} (b)^{2j+1} (x')^{2i_1} I_4(x') \quad (33) \end{aligned}$$

where

$$\begin{aligned} I_1(x') &= \int_{-a}^a (y')^{2n_2+\mu} \sqrt{b^2 + x'^2 + y'^2} dy' \\ &= \Pi_\xi (a)^{\xi-2t_1+1} (b^2 + x'^2)^{t_1-1} \sqrt{(a^2 + b^2 + x'^2)^3} + (-1)^{\xi/2} \frac{2(\xi-1)!!}{(\xi+2)!!} (b^2 + x'^2)^{\xi/2} \\ &\cdot (a\sqrt{a^2 + b^2 + x'^2} + (b^2 + x'^2) \operatorname{sh}^{-1} \frac{a}{\sqrt{b^2 + x'^2}}) \end{aligned}$$

$$\text{if } \mu \text{ is even} \quad (34a)$$

$$= 0 \quad \text{if } \mu \text{ is odd.} \quad (34b)$$

In the above expression

$$\pi_{\xi} = \frac{2}{v+1} \sum_{t_1=1}^{\xi/2} (-1)^{t_1+1} \frac{(\xi+1)(\xi-1)\dots(\xi+2t_1+5)}{(\xi+2)\xi\dots(\xi-2t_1+4)} \quad (35)$$

$$(\xi-1)!! = 1 \cdot 3 \dots (\xi-1) \quad \xi \text{ is even} \quad (36a)$$

$$(\xi+2)!! = 2 \cdot 4 \dots (\xi+2) \quad \xi \text{ is even} \quad (36b)$$

$$\xi = 2n_2 + \mu \quad (37)$$

The integral

$$I_2(x') = \int_{-a}^a (y')^{2i_2+\mu} \frac{dy'}{\sqrt{x'^2+y'^2}}$$

$$= \pi_{\xi_1}(a) \xi_1^{-2t_3+1} (x', 2)^{t_3-1} \sqrt{a^2+x'^2} + (-1)^{\xi_1/2} \frac{2(\xi_1-1)!!}{(\xi_1)!!} (x', 2)^{\xi_1/2} \text{sh}^{-1} \frac{a}{x'} \quad (38a)$$

$$= 0 \quad \text{if } \mu \text{ is odd.} \quad (38b)$$

where π_{ξ_1} is defined as the same as (32a), only if replace v with ξ_1 and

$$\xi_1 = 2i_2 + \mu \quad (39a)$$

$$(\xi_1-1)!! = 1 \cdot 3 \dots (\xi_1-1) \quad \xi_1 \text{ is even} \quad (39b)$$

$$(\xi_1)!! = 2 \cdot 4 \dots (\xi_1) \quad \xi_1 \text{ is even} \quad (39c)$$

The integral

$$I_3(x') = \int_{-a}^a \frac{(y')^{\xi_1}}{\sqrt{(x'^2+y'^2)^3}} dy'$$

$$= \pi'_{\xi_1}(a) \xi_1^{-2t_4+1} (x', 2)^{t_4-1} \sqrt{a^2+x'^2}$$

$$+ (-1)^{\frac{\xi_1-2}{2}} \frac{2(\xi_1-1)!!}{(\xi_1-2)!!} (x', 2)^{\frac{\xi_1}{2}-1} (\text{sh}^{-1} \frac{a}{x'} - \frac{a}{\sqrt{a^2+x'^2}})$$

$$= 0 \quad \text{if } \mu \text{ is even} \quad (40a)$$

$$= 0 \quad \text{if } \mu \text{ is odd.} \quad (40b)$$

where

$$\Pi'_{\xi_1} = \frac{2}{\xi_1+1} \sum_{t_4=1}^{\frac{\xi_1}{2}-1} (-1)^{t_4+1} \frac{(\xi_1+1)(\xi_1-1)\dots(\xi_1-2t_4+3)}{(\xi_1-2)(\xi_1-4)\dots(\xi_1-2t_4)} \quad (41)$$

The integral

$$\begin{aligned} I_4(x') &= \int_{-a}^a \frac{(y')^{\xi_1}}{\sqrt{(x'^2+y'^2)^{2j+1}}} dy' \\ &= \Pi''_{\xi_1} (a)^{\xi_1-2t_2+1} (x'^2)^{t_2-1} \frac{1}{\sqrt{(a^2+x'^2)^{2j-1}}} \\ &\quad + \sum_{t_0=1}^{j-1} (-1)^{\xi_1/2} \frac{2(\xi_1-1)!!(2)^{t_0}}{(\xi_1-2j)(\xi_1-2j-2)\dots(2-2j)} \frac{(j-1)(j-2)\dots(j-t_0)}{(2j-1)(2j-3)\dots(2j-2t_0+1)} \\ &\quad \cdot (a)(x'^2)^{\frac{\xi_1}{2}-t_0-1} \frac{(a^2+x'^2)^{t_0}}{\sqrt{(a^2+x'^2)^{2j-1}}} \\ &\quad + (-1)^{\xi_1/2} \frac{2(\xi_1-1)!!}{(\xi_1-2j)(\xi_1-2j-2)\dots(2-2j)(2j-1)} (a)(x'^2)^{\frac{\xi_1}{2}-1} \cdot \frac{1}{\sqrt{(a^2+x'^2)^{2j-1}}} \end{aligned}$$

$$= 0 \quad \text{if } \mu \text{ is even} \quad (42a)$$

$$= 0 \quad \text{if } \mu \text{ is odd} \quad (42b)$$

where

$$\Pi''_{\xi_1} = \frac{2}{\xi_1+1} \sum_{t_0=1}^{\xi_1/2} (-1)^{t_2+1} \frac{(\xi_1+1)(\xi_1-1)\dots(\xi_1-2t_2+3)}{(\xi_1-2j)(\xi_1-2j-2)\dots(\xi_1-2j-2t_2+2)} \quad (43)$$

(iii)

$$\begin{aligned}
& \iiint_{\Omega} (x')^{\lambda} (y')^{\mu} (z')^{\nu} \frac{dv'}{\sqrt{x'^2 + y'^2 + z'^2}} \\
&= \pi_{\nu} \pi_{\xi} \sum_{n_1, n_2} (-1)^{\nu/2} \frac{n!}{n_1! n_2!} (a)^{\xi - 2t_1 + 1} \cdot (b)^{\nu - 2t + 1} \cdot I_1 \\
&+ (-1)^{\xi/2} \pi_{\nu} \frac{2(\xi-1)!!}{(\xi+2)!!} (b)^{\nu - 2t + 1} \cdot I_2 \\
&+ \pi_{\xi_1} \sum_{i_1, i_2} (-1)^{\nu/2} \frac{i!}{i_1! i_2!} \frac{2(\nu-1)!!}{(\nu)!!} (a)^{\xi_1 - 2t_2 + 1} \cdot (b) \cdot I_3 \\
&+ \sum_{i_1, i_2} (-1)^{\frac{\nu + \xi_1}{2}} \frac{i!}{i_1! i_2!} \frac{2(\nu-1)!!}{(\nu)!!} \frac{2(\xi_1-1)!!}{(\xi_1)!!} (b) \cdot I_4 \\
&- \pi'_{\xi_1} \sum_{i_1, i_2} (-1)^{\nu/2} \frac{i!}{i_1! i_2!} \frac{2(\nu-1)!!}{3(\nu)!!} (a)^{\xi_1 - 2t_4 + 1} (b)^3 \cdot I_5 \\
&- \sum_{i_1, i_2} (-1)^{\frac{\xi_1 + \nu - 1}{2}} \frac{i!}{i_1! i_2!} \frac{2(\nu-1)!!}{3(\nu)!!} \frac{2(\xi_1-1)!!}{(\xi_1-2)!!} (b)^3 \cdot I_6 \\
&+ \pi'_{\xi_1} \sum_{i_1, i_2} \sum_{j=2}^{\infty} (-1)^{j + \frac{\nu}{2}} \frac{i!}{i_1! i_2!} \frac{2(\nu-1)!!}{(\nu)!!} \frac{(2j)!}{2^{2j} (j!)^2 (2j+1)} (a)^{\xi_1 - 2t_2 + 1} \cdot (b)^{2j+1} \cdot I_7 \\
&+ \sum_{i_1, i_2} \sum_{j=2}^{j-1} \sum_{t_0=1}^{t_0=1} (-1)^{j + \frac{\nu + \xi_1}{2}} \frac{i!}{i_1! i_2!} \frac{2(\nu-1)!!}{(\nu)!!} \frac{2(\xi_1-1)!!}{(\xi_1-2j)(\xi_1-2j-2)\dots(2-2j)} \\
&\cdot \frac{(2j)!}{2^{2j} (j!)^2 (2j+1)} \frac{(j-1)(j-2)\dots(j-t_0)}{(2j-1)(2j-3)\dots(2j-2t_0+1)} (a) (b)^{2j+1} \cdot I_8 \\
&+ \sum_{i_1, i_2} \sum_{j=2}^{\infty} (-1)^{j + \frac{\nu + \xi_1}{2}} \frac{i!}{i_1! i_2!} \frac{2(\nu-1)!!}{(\nu)!!} \frac{(2j)!}{2^{2j} (j!)^2 (2j+1)} \\
&\cdot \frac{2(\xi_1-1)!!}{(\xi_1-2j)(\xi_1-2j-2)\dots(2-2j)(2j-1)} (a) (b)^{2j+1} \cdot I_9
\end{aligned} \tag{44}$$

The integrals in (44) can be evaluated as in Appendix B.

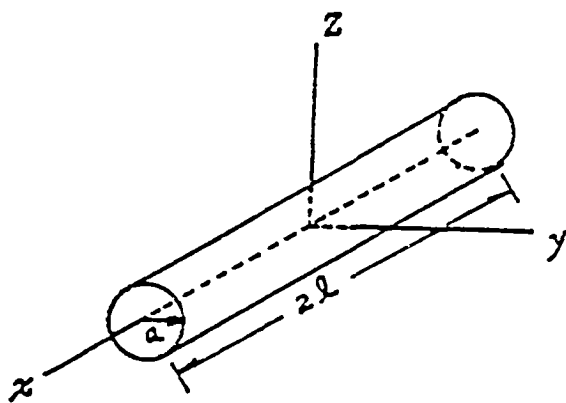


FIG. A1

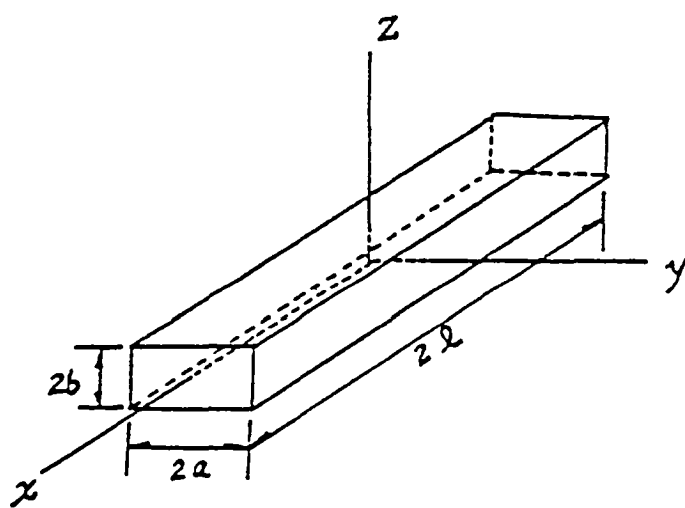


FIG. A2

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APPENDIX B

The I-Integrals and π -Functions

(a) The I-integrals

$$\begin{aligned}
 (i) \quad I_1 &= \int_{-\ell}^{\ell} (x')^{\lambda+2\eta} (b^2+x'^2)^{t_1-1} \sqrt{(a^2+b^2+x'^2)^5} \, dx' \\
 &= \pi_{\eta} \sum_{\Gamma_1, \Gamma_2} \frac{\Gamma_1!}{\Gamma_1! \Gamma_2!} (b)^{2\Gamma_1} (a^2+b^2)^{t_3-1} (\ell)^{\eta-2t_3+1} \sqrt{(a^2+b^2+\ell^2)^5} \\
 &\quad + \sum_{\Gamma_1, \Gamma_2} (-1)^{\eta/2} \frac{\Gamma_1!}{\Gamma_1! \Gamma_2!} \frac{8(\eta-1)!!}{(\eta+4)!!} (b)^{2\Gamma_1} (a^2+b^2)^{\eta/2} \left[\frac{1}{2} \ell \sqrt{(a^2+b^2+\ell^2)^5} + \right. \\
 &\quad \left. + \frac{3}{4} (a^2+b^2) (\ell \sqrt{a^2+b^2+\ell^2} + (a^2+b^2) \operatorname{sh}^{-1} \frac{\ell}{\sqrt{a^2+b^2}}) \right] \\
 &\quad \text{if } \lambda \text{ is even} \\
 &= 0 \quad \text{if } \lambda \text{ is odd}
 \end{aligned}$$

in which the π -function π_{η} is defined in (b) of the Appendix.

$$(ii) \quad I_2 = I_2' + I_2''$$

$$\begin{aligned}
 I_2' &= \int_{-\ell}^{\ell} a(x')^{\lambda+2\eta_1} (b^2+x'^2)^{t_3-1} \sqrt{a^2+b^2+x'^2} \, dx' \\
 &= \pi_{\eta_1} \sum_{\Theta_1, \Theta_2} \frac{\Theta_1!}{\Theta_1! \Theta_2!} (a)(b)^{2\Theta_1} (a^2+b^2)^{t_4-1} (\ell)^{\eta_1-2t_4+1} \sqrt{(a^2+b^2+\ell^2)^5} \\
 &\quad + \sum_{\Theta_1, \Theta_2} (-1)^{\eta_1/2} \frac{\Theta_1!}{\Theta_1! \Theta_2!} \frac{2(\eta_1-1)!!}{(\eta_1+2)!!} (a)(b)^{2\Theta_1} (a^2+b^2)^{\eta_1/2} \\
 &\quad \cdot (\ell \sqrt{a^2+b^2+\ell^2} + (a^2+b^2) \operatorname{sh}^{-1} \frac{\ell}{\sqrt{a^2+b^2}}) \\
 &\quad \text{if } \lambda \text{ is even} \\
 &= 0 \quad \text{if } \lambda \text{ is odd.}
 \end{aligned}$$

$$\begin{aligned}
I_2''' &= \int_{-\lambda}^{\lambda} (x')^{\lambda+2n_1} (b^2+x'^2)^{3\xi/2} \operatorname{sh}^{-1} \frac{a}{\sqrt{b^2+x'^2}} dx' \\
&= 2 \sum_{e_1, e_2} \frac{e!}{e_1! e_2!} \frac{a(b)^{2\theta_1(\ell)} \eta_3^{+1}}{\eta_3^{+1}} - \frac{1}{3} \sum_{e_1! e_2!} \frac{e!}{e_1! e_2!} \left(\sum_{t_6=1}^{\eta_3/2} (-1)^{t_6+1} \right. \\
&\quad \cdot \frac{(b^2)^{t_6-1}(\ell) \eta_3^{-2t_6+1}}{\eta_3^{-2t_6+1}} + (-1)^{\eta_3/2} (b)^{\eta_3-1} \cdot t_g^{-1} \frac{\ell}{b} (a)^3 (b)^{2e_1} + \\
&\quad + \sum_{e_1, e_2} \sum_{j_1=2}^{\infty} (-1)^{j_1} \frac{(2j_1)!}{2^{2j_1} (j_1)! (2j_1+1)} \frac{e!}{e_1! e_2!} (a)^{2j_1+1} \\
&\quad \cdot (\pi_{\eta_3} \frac{(b^2)^{t_7-1}(\ell) \eta_3^{-2t_7+1}}{(b^2+\ell^2)^{j_1-1}} + (-1)^{\eta_3/2} \frac{(\eta_3-1)!!}{(\eta_3-2j_1+1)(\eta_3-2j_1) \dots (3-2j_1)} \\
&\quad \cdot (b)^{\eta_3} (\pi_{\eta_3}'(\ell) + (-1)^{j_1-1} \frac{2}{b} t_g^{-1} \frac{\ell}{b})] (b)^{2e_1} \\
&\quad \text{if } \lambda \text{ is even} \\
&= 0 \quad \text{if } \lambda \text{ is odd.}
\end{aligned}$$

$$\begin{aligned}
\text{(iii)} \quad I_3 &= \int_{-\lambda}^{\lambda} (x')^{\lambda+2i_1+2t_3-2} \frac{1}{\sqrt{a^2+x'^2}} dx' \\
&= \pi_{\eta_2} (a^2)^{t_5-1} (\ell) \eta_2^{-2t_5+1} \frac{1}{\sqrt{(a^2+\ell^2)^3}} + (-1)^{\eta_2/2} \frac{2(\eta_2-1)!!}{(\eta_2+2)!!} (a^2)^{\eta_2/2} \\
&\quad \cdot (\ell \sqrt{a^2+\ell^2} + a^2 \operatorname{sh}^{-1} \frac{\ell}{a}) \quad \text{if } \lambda \text{ is even} \\
&= 0 \quad \text{if } \lambda \text{ is odd.}
\end{aligned}$$

$$\begin{aligned}
\text{(iv)} \quad I_4 &= \int_{-\lambda}^{\lambda} (x')^{\lambda+2i_1+\xi_1} \operatorname{sh}^{-1} \frac{a}{x'} dx' \\
&= \frac{a}{\eta_9+1} [\pi_{\eta_9} (a^2)^{t_{15}-1} (\ell) \eta_9^{-2t_{15}+1} \frac{1}{\sqrt{a^2+\ell^2}} + (-1)^{\eta_9/2} \frac{2(\eta_9-1)!!}{(\eta_9)!!} (a)^{\eta_9} \operatorname{sh}^{-1} \frac{\ell}{a}] \\
&\quad \text{if } \lambda \text{ is even} \\
&= 0 \quad \text{if } \lambda \text{ is odd.}
\end{aligned}$$

$$\begin{aligned}
(v) \quad I_5 &= \int_{-\ell}^{\ell} (x')^{\lambda+2i_1+2t_4-2} \frac{dx'}{\sqrt{a^2+x'^2}} \\
&= \Pi_{\eta_8}(a)^{t_{14}-1} \cdot (\ell)^{\eta_8-2t_{14}-1} \cdot \sqrt{a^2+\ell^2} + (-1)^{\eta_8/2} \frac{2(\eta_8-1)!!}{(\eta_8)!!} (a^2)^{\eta_8/2} \text{sh}^{-1} \frac{\ell}{a} \\
&\quad \text{if } \lambda \text{ is even} \\
&= 0 \quad \text{if } \lambda \text{ is odd.}
\end{aligned}$$

(vi)

$$I_6 = I'_6 - I''_6$$

$$\begin{aligned}
I'_6 &= \int_{-\ell}^{\ell} (x')^{\lambda+2i_1+\xi_1-2} \text{sh}^{-1} \frac{a}{x'} dx' \\
&= \frac{a}{\eta_{10}+1} [\Pi_{\eta_{10}}(a^2)^{t_{16}-1} \cdot (\ell)^{\eta_{16}-2t_{16}+1} \cdot \sqrt{a^2+\ell^2} + (-1)^{\eta_{10}/2} \frac{2(\eta_{10}-1)!!}{(\eta_{10})!!} \cdot \\
&\quad \cdot (a)^{\eta_{10}} \text{sh}^{-1} \frac{\ell}{a}] \quad \text{if } \lambda \text{ is even} \\
&= 0 \quad \text{if } \lambda \text{ is odd.}
\end{aligned}$$

$$\begin{aligned}
I''_6 &= \int_{-\ell}^{\ell} (x')^{\lambda+2i_1+\xi_1-2} \frac{a}{\sqrt{a^2+x'^2}} dx' \\
&= \Pi_{\eta_5}(a)^{t_9} \cdot (\ell)^{\eta_5-2t_9-1} \cdot \sqrt{a^2+\ell^2} + (-1)^{\eta_5/2} \frac{2(\eta_5-1)!!}{(\eta_5)!!} (a)^{\eta_5/2} \text{sh}^{-1} \frac{\ell}{a} \\
&\quad \text{if } \lambda \text{ is even} \\
&= 0 \quad \text{if } \lambda \text{ is odd.}
\end{aligned}$$

$$\begin{aligned}
(vii) \quad I_7 &= \int_{-\ell}^{\ell} (x')^{\lambda+2i_1+2t_2-2} \frac{dx'}{\sqrt{(a^2+x'^2)^{2j-1}}} \\
&= \Pi_{\eta_6}(a^2)^{t_{10}-1} \frac{(\ell)^{\eta_6-2t_{10}+1}}{\sqrt{(a^2+\ell^2)^{2j-3}}} + (-1)^{\eta_6/2} \frac{2(\eta_6-1)!!}{(\eta_6-2j-2)(\eta_6-2j-4)\dots(-2j)}
\end{aligned}$$

$$\frac{a^{\eta_6} \ell}{(2j-3) a^2 \sqrt{(a^2 + \ell^2)^{2j-3}}} \left(1 + \sum_{k_{11}=1}^{j-2} \frac{8^{k_{11}} (j-2)(j-3) \dots (j-k_{11}-1) (a^2 + \ell^2)^{k_{11}}}{(2j-5)(2j-7) \dots (2j-2k_{11}-3) (4a^2)^{k_{11}}} \right)$$

if λ is even

if λ is odd.

= 0

(viii)

$$I_8 = \int_{-\ell}^{\ell} (x')^{\lambda+2i_1+\xi_1-2t_0-2} \frac{(a^2+x'^2)^{t_0} dx'}{\sqrt{(a^2+x'^2)^{2j-1}}}$$

$$= \pi_{\eta_7} \sum_{\phi_1, \phi_2} \frac{\phi_1! \phi_2!}{\phi_1! \phi_2!} \frac{(a^2)^{t_{12}-1+\phi_1} \ell^{\eta_7-2t_{12}+1}}{\sqrt{(a^2+\ell^2)^{2j-3}}} + \sum_{\phi_1, \phi_2} \frac{\phi_1! \phi_2!}{\phi_1! \phi_2!}$$

$$\cdot (-1)^{\eta_7/2} \frac{2(\eta_7-1)!!}{(\eta_7-2j-2)(\eta_7-2j-4) \dots (-2j)}$$

$$\frac{(a^2)^{\eta_7/2+\phi_1} \ell}{(2j-3) a^2 \sqrt{(a^2+\ell^2)^{2j-3}}} \left(1 + \sum_{k_{13}=1}^{j-2} \frac{8^{k_{13}} (j-2)(j-3) \dots (j-k_{13}-1) (a^2+\ell^2)^{k_{13}}}{(2j-5)(2j-7) \dots (2j-2k_{13}-3) (4a^2)^{k_{13}}} \right)$$

if λ is even

if λ is odd.

= 0

(ix)

$$I_9 = \int_{-\ell}^{\ell} (x')^{\lambda+2i_1+\xi_1-2} \frac{dx'}{\sqrt{(a^2+x'^2)^{2j-1}}}$$

$$= \pi_{\eta_8} \frac{(a^2)^{t_{12}-1} \ell^{\eta_8-2t_{12}+1}}{\sqrt{(a^2+\ell^2)^{2j-1}}} + (-1)^{\eta_8/2} \frac{2(\eta_8-1)!!}{(\eta_8-2j-2)(\eta_8-2j-4) \dots (-2j)}$$

$$\frac{a^{\eta_8} \ell}{(2j-3) a^2 \sqrt{(a^2+\ell^2)^{2j-3}}} \left(1 + \sum_{k_{12}=1}^{j-2} \frac{8^{k_{12}} (j-2)(j-3) \dots (j-k_{12}-1) (a^2+\ell^2)^{k_{12}}}{(2j-5)(2j-7) \dots (2j-2k_{12}-3) (4a^2)^{k_{12}}} \right)$$

if λ is even

if λ is odd.

= 0

(b) The π -Functions

The π -functions in the I-integrals listed in (a) of this Appendix are defined as follows:

$$(i) \quad \pi_n = \frac{2}{n+1} \sum_{t_3=1}^{n/2} (-1)^{t_3+1} \frac{(n+1)(n-1)\dots(n-2t_3+3)}{(n+4)(n+2)\dots(n-2t_3+6)}$$

$$n = \lambda + 2n_2 + 2\Gamma_2 \quad (\text{is even})$$

$$\Gamma_1 + \Gamma_2 = \Gamma = t_1 - 1$$

$$(ii) \quad \pi_{n_1} \text{ as the same as (35), if replace } \xi \text{ with } n_1$$

$$n_1 = \lambda + 2n_1 + 2\theta_2 \quad (\text{is even})$$

$$\theta_1 + \theta_2 = \theta = \xi/2$$

$$(iii) \quad \pi_{n_3} = \frac{2}{n_3+1} \sum_{t_7=1}^{n_7/2} (-1)^{t_7+1} \frac{(n_3+1)(n_3-1)\dots(n_3-2t_7+3)}{(n_3-2j_1+1)(n_3-2j_1-1)\dots(n_3-2j_1-2t_7+3)}$$

$$\pi'_{n_3} = \frac{4}{3-2(j_1+1)} \sum_{t_8=1}^{j_1-1} (-1)^{t_8+1} \cdot (2)^{t_8-1} \cdot \frac{[3-2(j_1+1)][3-2j_1]\dots[3-2(j_1-t_8+2)]}{(j_1-1)(j_1-2)\dots(j_1-t_8)}$$

$$n_3 = \lambda + 2n_1 + 2e_2 \quad (\text{is even})$$

$$e_1 + e_2 = e = \frac{3}{2} \xi$$

$$(iv) \quad \pi_{n_2} \text{ as the same as (35), if replace } \xi \text{ with } n_2$$

$$n_2 = \lambda + 2i_1 + 2t_3 - 2 \quad (\text{is even})$$

$$(v) \quad \pi_{n_9} \text{ as the same as (32a), if replace } v \text{ with } n_9$$

$$n_9 = \lambda + 2i_1 + \xi_1 \quad (\text{is even})$$

(vi) π_{η_8} as the same as (32a), if replace v with η_8

$$\eta_8 = \lambda + 2i_1 + 2t_4 - 2 \quad (\text{is even})$$

(vii) $\pi_{\eta_{10}}$ as the same as (32a), if replace v with η_{10}

$$\eta_{10} = \lambda + 2i_1 + \xi_1 - 2 \quad (\text{is even})$$

(viii) π_{η_5} as the same as (32a), if replace v with η_5

$$\eta_5 = \lambda + 2i_1 + \xi_1 - 2 \quad (\text{is even})$$

$$(ix) \quad \pi_{\eta_6} = \frac{2}{\eta_6+1} \sum_{t_{10}=1}^{\eta_6/2} (-1)^{t_{10}+1} \frac{(\eta_6+1)(\eta_6-1)\dots(\eta_6-2t_{10}+3)}{(\eta_6-2j-2)(\eta_6-2j-4)\dots(\eta_6-2j-2t_{10})}$$

$$\eta_6 = \lambda + 2i_1 + 2t_2 - 2 \quad (\text{is even})$$

(x) π_{η_7} as the same as π_{η_6} , if replace η_6 with η_7

$$\eta_7 = \lambda + 2i_1 + \xi_1 + 2\phi_2 - 2t_0 - 2 \quad (\text{is even})$$

$$\phi_1 + \phi_2 = \phi = t_0$$

(xi) π_{η_8} as the same as π_{η_6} , if replace η_6 with η_8

$$\eta_8 = \lambda + 2i_1 + \xi_1 - 2 \quad (\text{is even})$$

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16. Abstract This work identifies a foundation for ultrasonic evaluation of microcrack nucleation mechanics. The objective is to establish a basis for correlations between plane strain fracture toughness and ultrasonic factors through the interaction of elastic waves with material microstructures, e.g., grain size or second-phase particle spacing. Since microcracking is the origin of (brittle) fracture it is appropriate to consider the role of stress waves in the dynamics of microcracking. Therefore, this work deals with the following topics: (1) microstress distributions with typical microstructural defects located in the stress field, (2) elastic wave scattering from various idealized defects, (3) dynamic effective-properties of media with randomly distributed inhomogeneities.			
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